THIS book, by an author who has many years' experience in teaching geometrical optics, provides a modern treatment of the subject suitable for Honours Physics courses. It aims at establishing a clear relation between Geometrical Optics and the relevant parts of Physical Optics, and other branches of physics.

While it seeks to avoid being too academic and mathematical, an effort is made to exclude the uncertain approximations found both in older books and in some more recent treatments.

The book should provide a sound foundation for students proceeding to specialize in Optical Sciences or related subjects, and demands only a modest mathematical equipment in analytical geometry and elementary calculus.

A chapter is included on the fundamental theorems of photometry and their relation to practical problems.

Each chapter is accompanied by a set of exercises.
GEOMETRICAL OPTICS

By

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Preface

The subject of Geometrical Optics continues to form an essential part of University Courses in Physics. The interest is, no doubt, partly utilitarian, since optical instruments constitute an essential part of modern scientific and technical equipment. On the other hand the subject has an intellectual discipline of its own; it is essential for the proper understanding of physical optics and it has, in its wider aspects, intimate connexions with other branches of physics.

The traditional academic treatment of the subject in England, developed in such elegance by the Cambridge School and others, makes a demand upon the time and mathematical equipment of the student which cannot usually be afforded by those reading for a Physics Degree. Alternative presentations have been attempted, but they have often created an unpleasant impression: the discussion is felt to be approximate and unsatisfactory if no real attempt is made to ascertain the limits within which the treatment is valid; and the student is finally left wondering how real optical problems can possibly be tackled. Even the "classical" texts are by no means free from objections of the same kind. At the same time the ever-increasing scope of Physics makes it essential to restrict the field of any such subject to the minimum essentials; and brevity is important.

The present writer has had a considerable experience in teaching Technical Optics and has also had, for many years, the responsibility of introducing the subject of Geometrical Optics to an Honours Degree Class in Physics.

The general plan suggested by experience is embodied in the text of the present book, though it is naturally more comprehensive than the scope of the usual short course of lectures. It should, however, lend itself to the selection of a group of topics for study.

The treatment aims at early confidence in the subject by the use of exact results, with a discussion of the degree of approximation when problems relating to spherical surfaces have to be faced. The first chapter is wide in its scope, and aims at the achievement of a grounding in some of the most important topics and techniques of discussion; so that the latter sections of the book should not be found to present ever-increasing difficulties.

The analytical discussion of collinear systems has been adopted
rather than Maxwell's rather long geometrical presentation. Although the "optical" applicability of the collinear theory has to be taken with reservation, it is difficult to see how the broad phenomena of optical images can be effectively presented without some such framework of terminology.

In the higher stages of optical theory, into which this book does not enter, it is quite true that the full significance of many general optical theorems is most readily appreciated from a proper analytical treatment. For example the optical sine theorem is a special case of the more general cosine relation. There is good reason to believe that the difficulties of higher aberration theories can best be overcome by analytical discussions in which the "diagrams," so helpful in gaining physical ideas, are perforce discarded. But the physicist needs to appreciate his instruments in corporeal terms, and his pictures of aberrations show unwanted kinks in wave fronts. When his ideas are properly grounded he can more safely use the necessary formulae. The present book makes no claim, then, to give a fully developed treatment of the subject, but on the other hand some efforts have been made to avoid the loose and slipshod arguments which have often marred the presentation of the subject to beginners, and to pave the way for more detailed studies. With regard to the need for brevity, space has been saved by treating reflection as a special case of refraction. Moreover, once the student has mastered the derivation of various formulae for the refracting system, the proofs of the reflection formulae afford simple riders to illustrate similar methods.

With regard to the discussion of optical instruments, it seems that the Physics Degree student will not have time to study more than the fundamental points of telescopes, microscopes, spectrosopes, and photometers. Problems incidental to the subject of photographic lenses are discussed throughout the text.

Instruments like sextants, range-finders, refractometers, and many more could receive only trivial treatment in any book of this kind, and as the real optical problems arising in connexion with them are numerous and difficult, it seems better to omit them altogether. Some of them are briefly treated in introductory books on Light and, failing a thorough discussion, that is deemed sufficient.

I am greatly indebted to colleagues of the Imperial College, especially Professor W. D. Wright and Dr. W. Weinstein, for criticism and helpful discussion of parts of the text.
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CHAPTER I

The Laws of Geometrical Optics

Optics is the science of vision, and all that pertains thereto, including instruments and appliances used as aids to sight. The term is derived from the Greek word ὄπτωκη (Optics) which refers both to vision and the objective phenomena of light.

Most of the early developments of the subject were based upon the idea that light consists of "rays." The Latin word radius, of which one meaning is a spoke (of a wheel), was used to denote also a ray from the sun, so that the divergence of rays in straight lines from a source of light is thus implied.

Geometrical Optics begins with the geometrical and analytical study of the relative paths of rays. Some of the fundamental inventions of optical instruments preceded the formulation of the laws of this subject; for example the earliest production of glass lenses is lost in antiquity; but the perfection of instruments has only been possible by the use of the exact science. Achromatic lenses could not be made satisfactorily, as will be explained, until the quantitative facts regarding the dispersion of light by various glasses had been ascertained, as by Fraunhofer (about A.D. 1814).

Rays

The usual physical concept behind the geometrical idea of rays was that they were the paths of flying corpuscles; but when the corpuscular theory of light, in its early form, was shown to be untenable the wave theory took its place, and has been proved adequate to explain practically all the phenomena of importance in connexion with ordinary optical instruments. Many recent improvements in the design of optical instruments have been effected only through the application of ideas based on the wave theory. Thus while the elementary laws of geometrical optics carry the subject to a certain stage of development, further progress needs to superimpose on these the ideas of the vibratory or wave nature of light, and the "optical paths" of the coherent oscillatory disturbances which may travel outwards from an elementary source of light. These "optical paths" determine the relative phases of the disturbances arriving in a focus and the neighbouring regions.
The relations between geometrical and physical optics are discussed in Chapter IV, though this question, and the various theories of light, are more fully treated in text-books on Physical Optics.* Quantum phenomena will not be discussed here. The geometrical laws of reflection and refraction, together with a few assumptions about the effects of rays, are sufficient to treat the elementary theory of the common optical instruments which are the subjects of Chapters I, II, and III. However, in the first instance the concept of the ray should be more closely examined.

The following experiment can be performed, as illustrated in Fig. 1. The source of light $L$ (a candle or an electric lamp with a diffusive bulb would be suitable) sends light through a small hole in an opaque screen $A$. An inverted image of the source is seen on the white diffusely reflecting plane screen $B$.† As nearly as can be measured, it will be found that the ratio of the sizes of the image and the source is equal to the ratio of their respective distances from the pinhole. Using the symbols shown in the diagram, the above relation is expressed by

$$\frac{k'}{k} = \frac{l'}{l}$$

(It is sufficient for the present to treat all these linear magnitudes as positive numbers.)

**Rectilinear Propagation**

The phenomenon described in the section above, which was first clearly described by Della Porta in 1558 in his book *Magia Naturalis*, was explained by him as a consequence of the rectilinear propagation

† The lens can be removed from a camera and replaced by an opaque disc pierced by a pinhole. The image (somewhat faint) will be seen on the focusing screen if extraneous light is excluded.
of light (rays travel in straight lines in a homogeneous medium). A ray can be assumed to start from any infinitesimal part of the source, travel through the hole and reach the screen without deflexion.

Now if the hole in screen A is enlarged, the image on B becomes brighter, but at the same time less sharply defined. These two results are very significant.

Note first that if the pinhole in A has a finite size, light can reach a point such as P in the image from a finite region in the source; a cone with its apex in P and its base constituted by the pinhole can be produced to intersect the source; all parts of the source within

the cone can send rays to the point P; and if the size of the pinhole is increased it increases the magnitude of the region of the source contributing light to any such point. This is clearly consistent with the probability of an increased total number of rays reaching P.

Now if we assume that each ray is independent of the others, in its path and effectiveness, the increased brightness can be associated with a greater number of rays.

Secondly; rays emerging in various directions from any one point Q of the source can reach the screen within a similar cone with its apex in Q, produced till it meets the screen B. Therefore the point in the object is represented, not by one point in the image, but by a finite area. The image is therefore not sharp, but will be the more ill-defined the larger the pinhole.

Consider the particular case of a source which is a short luminous line (LM, Fig. 2) symmetrically placed with respect to the axial direction of a circular pinhole and parallel to both the screens, which are themselves normal to the axis. The overlapping circular patches of illumination which would be found on the receiving screen if the source were a line of discrete points are suggested in the figure. However, if the source is uniform, no discrete patches will be found,
but a long patch of light shaded towards its sides and ends. The distances between source and screens, etc., are as in Fig. 1; the diameter of the pinhole is $d$. If the source is uniform, it is reasonable (in accordance with the ideas formulated above) to suppose that the illumination at any point $P$ on the receiving screen is closely connected with the length of the source appearing within the imaginary cone (produced) of which the apex is $P$ and the base is defined by the hole $H$. Considering the point $P$ to move along the mid-line of the image patch as suggested by the arrow in the figure, the illumination begins when this cone just includes the end $L$ of the source. After a further inward movement (see Fig. 3) of $d(l + l')/l$, the imaginary cone contains the maximum possible length of the line source and the maximum illumination will be expected; there should be a continuous increase over the region of transition. From the central region, the illumination decreases also in a perpendicular direction. The length of the line source comprised within the cone can be written down by reference to Fig. 4. Note that the radius of the cone in the plane containing the source is $\frac{1}{2}d(l + l')/l'$, and if $h$

![Fig. 3.](image)

![Fig. 4.](image)

is the upward displacement of the point (Q) under consideration from the central line of maximum illumination in the image, the centre of the circular intercept by the cone in the source plane is at a distance $h(l/l')$ below the line of the source. We have therefore to write down the length $c$ of the chord in a circle of known radius at a given distance from centre, i.e.

$$c = 2\left(\frac{d^2}{4} \left(\frac{l'}{l'} + \frac{l}{l'}\right)^2 - h^2 \left(\frac{l}{l'}\right)^2\right)^{\frac{1}{2}}$$
The illumination should fall off slowly at first and drop to zero when

\[ h = \frac{d}{2} \left( \frac{l' + l}{l} \right) \]

If \( h \) exceeds this value the illumination would disappear, the algebraic result being imaginary. These expectations are found to accord closely with practical observations. Thus a theory based on the above supposed properties of rays will account broadly for the two significant results mentioned above, though the discussion has not formulated any exact photometric definitions. Its limitations in that respect will appear after the discussions in Chapter VI.

It would perhaps appear that the isolation of single rays might be feasible if the size of the source and pinhole were further and further decreased. Attempts to do this, however, using a very small bright source, show that the reduction of the size of the pinhole beyond a certain point results in an actual progressive \textit{enlargement} of the illuminated region on the screen, together with a corresponding reduction of the illumination. The reasons for this are discussed by theories of Physical Optics. However, for a modest size of aperture, the patch of light remains fairly well defined; if the source is very small and the pinhole is circular the patch has a circular symmetry and may consist of a disc bordered by rings of light (diffraction; see below); the pattern may change according to the distance of the screen from the pinhole. However, we may visualize the locus of the centre of the pattern as representing the path of a ray.

The broad features of the phenomena of ordinary shadows are also explained on similar lines. If the source is small enough, the outline of the shadow of an opaque object on a plane screen is very closely represented by its geometrical projection with respect to the “point” source. If the source is large, its component elements all contribute their projections, so that the border of the shadow is increasingly diffuse as the source becomes larger.

If there is any point on the screen not reached by a straight ray from the source, it lies in the region of total shadow. Points which can be reached by rays from a part only of the source are in the region of partial shadow or \textit{penumbra} (from the Latin \textit{paene} (almost), and \textit{umbra} (shadow)). The calculation of the illumination in the penumbral region will involve considerations similar to those above, but will also need the fuller formulation of photometric theory. In detailed studies it is found that owing to the wave-nature of light there is some very slight degree of illumination within the geometrical border of a shadow. This phenomenon is an aspect of “diffraction” and is again discussed in the theory of Physical Optics.
Reflection and Refraction

When rays of light are incident at a surface separating two differing transparent media, some of the light is, in general, reflected* back into the original medium, and some is refracted† across the boundary into the second medium.

If the surface is optically polished (or without appreciable roughness, as in the case of liquid surfaces), the reflection and refraction are generally accompanied by abrupt changes of direction of the rays according to ascertainable laws. Diffuse reflection and refraction is partly explicable in terms of the multiplicity of directions of the small facets or surface elements which constitute an unpolished surface.

Owing to the difficulty of obtaining sufficient light from very small sources, and also partly to diffraction, the task of the early investigators who strove to determine the exact laws of the reflection and refraction of rays was by no means easy. The full history of these attempts must be studied elsewhere, but a typical method described in Kepler's† Dioptrics (A.D. 1611) is illustrated in Fig. 5. A horizontal board is fitted with a vertical end-piece terminated by a straight horizontal edge CBD. Sunlight throws a shadow on the board, which is furnished with a scale indicating distances from the end. A rectangular glass block with polished faces can be placed adjacent to the vertical end piece. The observation of the shadows IQ and HK, within the block and outside it respectively, furnishes quantitative data for the study of refraction.

* From the Latin reflecto (I turn back).
† From the Latin re (again) and fractus (broken).
‡ Kepler's experiments on Optics were made at the Observatory, Prague. He did not succeed in finding the precise law of refraction.
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If the horizontal board is furnished with two end pieces, and a plane mirror is placed horizontally on the base between them, the laws of reflection may conveniently be studied.

The law of reflection (stated below) was known to the Greeks. The exact law of refraction was found (but not published) by W. Snell* (1591–1626) and put into analytical form by the famous mathematician-philosopher Descartes (Dioptrique, 1637). In modern times the laws can be verified by much more precise methods.

It is convenient to define the angle of incidence as the acute angle (at the point of incidence) between the ray and the normal to the surface; the angles of reflection and refraction are likewise the acute angles (at the point of incidence) between the reflected and refracted rays respectively and the normal.

The plane of incidence is the plane containing the incident ray (at the point of incidence) and the normal.

The law of reflection may be stated in two parts thus—

(a) The angles of incidence and reflection are equal.
(b) The reflected ray lies in the plane of incidence.

The law of refraction is similarly presented—

(a) The sine of the angle of incidence bears a constant ratio to the sine of the angle of refraction.
(b) The refracted ray lies in the plane of incidence.

The above laws apply, strictly, only to media of which the refractive properties do not depend either on the direction of the ray nor the direction of the vibrations in the light,† and to light of some specified frequency or wavelength. The phenomena of dispersion are discussed in books on physical optics; they are due to the differences of velocity with which light is propagated, according to the frequency, in any given medium. However, for a homogeneous medium precisely defined in physical terms and a given frequency of light there is a unique value of the velocity. An immediate consequence of this can be shown to be the reversibility of ray paths; light can travel in either direction along a ray path through an optical system.

From another point of view, the principle of reversibility can be quoted as an experimental datum; it leads to the possibility of assigning a refractive index to a given medium for a given frequency. This term must now be explained. Considering the path of a ray through a plane parallel plate of glass in air, Fig. 6(a), in which the plane of incidence is supposed to be the plane of the paper, let the

* Professor of Mathematics in the University of Leyden.
† i.e. isotropic media; the refractive properties are "scalar" quantities.
angles of incidence and refraction at the top surface be \( i \) and \( i' \) respectively. The law of refraction as quoted above gives

\[
\frac{\sin i}{\sin \hat{i}} = \text{constant (say } n') \quad \cdots \quad (1.00)
\]

The angle of incidence on the lower face is \( i'' \). If \( i'' \) be the angle of refraction into air again, the law of refraction would entail

\[
\frac{\sin i'}{\sin i''} = \text{constant}
\]

but in order that the ray path may be reversible, \( i'' \) must be equal to \( i \); for otherwise the condition of refraction would differ at the two surfaces, and there is no reason to suppose that this could be the case. Hence the law of refraction at the second surface must be

\[
\frac{\sin i'}{\sin i} = \text{constant} = \frac{1}{n'} \quad \cdots \quad (1.01)
\]

and then the ray emerges parallel to its original direction.

Consider now a case in which the ray passes through two separated plane parallel slabs with adjacent faces parallel (Fig. 6(b)). Let the ratio of the sines of the angles \( i \) and \( i' \) in the top slab be \( n' \) (exactly as above); and in the bottom slab be \( n_1 \). By an exactly similar argument, the ray will enter and leave the second slab with an initial angle of incidence and final angle of refraction both equal to \( i \); however, the angle of refraction inside its upper face will be \( i_1 \) (say), where

\[
\frac{\sin i}{\sin i_1} = n_1 \quad \cdots \quad (1.02)
\]
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If now the two slabs are brought together, the air space between them will ultimately disappear, but there is no reason to suppose that this could cause any sudden change in the angles of the rays. The angle of incidence at the face of contact in the first slab is now \( i' \), and the angle of refraction in the second slab is \( i_1 \). The law of refraction at the interface is given by multiplying together corresponding sides of equations (1.01) and (1.02) above, thus obtaining

\[
\frac{\sin i'}{\sin i_1} = \frac{n_1}{n'}
\]

or, written symmetrically,

\[
n_1 \sin i_1 = n' \sin i'
\]

This form of the law is generally applicable. The equation (1.00) above is the particular case in which the "\( n \)" corresponding to the air is unity; the value of \( n' \) is then the refractive index of the medium given with respect to the air. Similarly the \( n_1 \) in equation (1.02) is the refractive index of the second slab, also with respect to air. The general form of the law is usually written

\[
n \sin i = n' \sin i'
\]

where the unaccented quantities refer to the medium of incidence.

Tables of refractive indices (see Appendix III) of particular kinds of glass, etc., usually quote the values with respect to dry air at 20°C, and 760 mm pressure.

Actually, the refractive index of air itself with respect to a vacuum (assumed refractive index unity) is about 1.00029, and the variations of pressure, humidity, etc., will also cause variations of its refractive index, but these changes can usually be neglected in connexion with the performance of optical instruments.

Liquids and (to a lesser extent) solids generally exhibit appreciable temperature changes in refractive index (a rise of 1°C may easily affect the value of the third decimal place in the figures for a liquid) so that temperature control is essential during precise measurements.

The accuracy usually required in absolute measurements of refractive indices of glasses for optical calculations should be sufficient to give the result within one or two units in the fourth decimal place. Differences of refractive index (needed in dispersion measurements) can often be found by suitable experimental methods, within one or two units of the fifth decimal place, even if the absolute value is known less accurately. The requirements will be more easily understood after the discussion in Chapter III.

Optical glass is produced with a very high degree of homogeneity,
so that local variations of refractive index should only correspond to changes of one or two units in the fifth decimal place of the numerical value.

The Deviation of a Ray

The deviation $\Delta$ (see Fig. 7) of a ray on refraction is clearly equal to the difference of the angles of incidence and refraction

$$\Delta = i - i'$$

and a change of deviation $d\Delta = di - di'$.

From the law of refraction we find by taking logarithms

$$\log n + \log \sin i = \log n' + \log \sin i'$$

whence by differentiating with respect to $i$,

$$\frac{di}{\tan i} = \frac{di'}{\tan i'} \quad . \quad (1.04)$$

Hence if $n'$ is greater than $n$ so that $i'$ is less than $i$, $di'$ will be less than $di$. Therefore the deviation increases with the angle of incidence.

Owing to the principle of reversibility, the deviation also increases with the angle of incidence if the ray proceeds to a medium in which $n'$ is less than $n$, and the maximum possible deviation occurs, of course, when the refracted ray emerges at $90^\circ$.

The above equation (1.04) can be written in the form

$$\frac{di}{di'} = \frac{n' \cos i'}{n \cos i}$$

Whence

$$\left( \frac{di}{di'} \right)^2 = \frac{n'^2 (1 - \sin^2 i')}{n^2 \cos^2 i} = \frac{n'^2 - n^2 \sin^2 i}{n^2 \cos^2 i}$$

$$= 1 + \frac{n'^2 - n^2}{n^2 \cos^2 i} \quad . \quad . \quad . \quad . \quad (1.05)$$

Hence $di/di'$ increases with the growth of $i$. Thus the deviation of a ray shows an accelerated rate of increase as $i$ continues to increase.

Heath's Laws

There are several relations between the paths of incident and refracted ray which are useful in the theory of instruments, and the
type of argument which follows was used by Heath* in proving them.

Referring to Fig. 8, let O be the origin of a Cartesian system of coordinates, and let a ray AO be incident at a refracting surface coinciding with the XZ plane; its path after refraction is OB. Take the lengths OA and OB so that they represent the indices of refraction $n$ and $n'$ of the homogeneous media traversed before and after refraction.

Now the normal at O is the axis OY, and if perpendiculars AM and BN are drawn to the normal from A and B respectively, the law of refraction gives

$$\text{AM} = \text{BN}$$

and requires that these two intercepts are parallel since they lie in one plane with the normal.

Again, drop perpendiculars AP and BQ from A and B respectively to the YZ plane. Since the angle $\hat{MAP} = \hat{NBQ}$, it is clear that the triangles MAP and NBQ are equal in all respects.

Now let the angle $\hat{AOP}$ between the ray and its projection on the YZ plane be called $\eta$, while the angle $\hat{POM}$ between this projection and the normal is $\phi$; the addition of an accent indicates the corresponding quantities for the refracted ray. Since $\text{AP} = \text{BQ}$ we now have

$$n \sin \eta = n' \sin \eta' \quad \text{.} \quad (1.06)$$

Further since $\text{MP} = \text{NQ}$ we have

$$\text{OP} \sin \phi = \text{OQ} \sin \phi'$$

which gives

$$n \cos \eta \sin \phi = n' \cos \eta' \sin \phi' \quad \text{.} \quad (1.07)$$

We may further note that MP is equal to the projection of OA on the OZ axis and QN equal to the projection of OB. Hence if the

* The relations are very quickly obtained from the vector form of the law of refraction (Appendix I).
acute angles $\hat{A}OZ$ and $\hat{B}OZ'$ are written $\alpha$ and $\alpha'$ respectively, we have

$$n \cos \alpha = n' \cos \alpha' \quad (1.08)$$

The above equations are quite independent of the position of the ray relative to the axis, and thus these equations (1.06) and (1.07) relate to projection on any plane containing the normal, while (1.08) holds for the angles between the ray and any straight line tangent to the surface at the point of incidence.

**Total Reflection and the Critical Angle**

Applying the law of refraction to the passage from a denser ($n$) to a rarer medium ($n'$) we can write it

$$\sin i' = \frac{n}{n'} \sin i$$

Since $n/n'$ is greater than one, it is evident that $\sin i$ must not exceed $n'/n$ if $i'$ is to be real, since the sine of an angle cannot exceed unity. When $\sin i = n'/n$, the emergent ray grazes the surface; the angle of incidence is then the "critical angle." Should the angle $i$ further increase, no ray emerges into the rarer medium. Observation shows that as the angle increases from low values the amount of the reflected light shows a continuous increase, but there is a comparatively sudden rise at the critical angle. The light is then said to be "totally reflected."

The phenomenon is of much importance in connexion with the design of erecting prisms (see p. 92 below). The higher the refractive index of a medium, the smaller is the critical angle for emerging rays.

**Real Images and Their Properties**

The image in the pinhole camera, discussed above, is an example of a "real" image. It can be received upon a screen and thus made visible to the eye; or it is capable of being recorded by a photographic plate.

As already mentioned, rays can be imagined to originate from any infinitesimal element of a source of light (or of a body which is diffusely reflecting or scattering light). In general, such rays spread in all directions, and it might be supposed that if the luminous intensity increased there would be a greater number of rays. The illumination of the pinhole image could be increased when more rays could be received at a given point of the screen from a larger area of the source. If some means were available, however, to bring together, in one point of the image, more and more of the rays
diverging from one point of the object we could expect to increase the illumination of the image without loss of definition.

It is shown in analytical geometry* that the lines joining any point B of an ellipse (Fig. 9) to the two foci (H and I) make equal angles with the normal BN to the curve at the point in question. The same must be true for an ellipsoid of revolution (or prolate spheroid) made by rotating the ellipse around the axis through the two foci.

Accordingly, if a point source of light is placed at one focus of an ellipsoidal reflector, all the rays will be re-united at the other focus. A familiar example of a special case of this action is the paraboloidal reflector of a searchlight. The paraboloid is the special case of an ellipsoid of revolution with one focus at an infinite distance. The rays from the small source (perhaps a small arc-lamp) at the near focus are all directed towards the far image. The arc has a finite size, but the rays from any element of it truly at the focus would be rendered parallel on reflection if the paraboloid were perfect. Ellipsoidal reflectors are less familiar, but they are used in cinematograph projectors; they produce a bright “real image” of the source, and the illumination increases with the size of the mirror. In the case of these reflectors, it is a well known geometrical property that the sum of the distances from any point on the surface to the two foci is constant. Hence the light travelling from one focus to the other would always take exactly the same time in transit if the velocity is independent of the direction.

Paraboloidal mirrors are often used in astronomical telescopes; the source of light (a star) is so far away that its distance can be regarded as "infinite" for purposes of optical calculations; the real image at the principal focus can be received on a photographic plate.

The theory of the action of mirrors or mirror systems was called *Catoptrics* by Euclid (Greek κατοπτρον, a mirror). As against this, *Dioptics* was the optics of the behaviour of light in passing through transparent optical media (Greek δια, through). In particular, it is concerned with the action of lenses and refracting systems.

The trigonometrical form of the law of refraction was used by Descartes in the problem of finding the exact form of a refracting surface which will re-unite all the rays, diverging from a point source, into a point focus. He first pointed out the special properties of ellipsoidal and hyperboloidal surfaces. The following is a much shortened trigonometrical form of his geometrical discussion.

Fig. 9 shows a principal section DBK of an ellipsoid of revolution with foci H and I, and a ray AB parallel to the major axis is incident at B. We now suppose that the ellipsoid is the surface of a homogeneous *refracting* body. The normal CBN, cutting this axis in N, bisects the angle $\hat{AB}$; hence $\hat{ABC}$ is the angle of incidence $i$. We shall seek the condition that $\hat{NBI}$ may be the angle of refraction.

Let $\hat{NBI}$ be written $r$. Considering the triangle BNI, the exterior angle $\hat{HN}B$ is clearly equal to $i$, and thus

$$\frac{\sin i}{\sin r} = \frac{BI}{NI}$$

Now if we draw a line through H parallel to NC, and cutting IB (produced) in Q, then $\hat{BQH} = \hat{IBN} = r$, and $\hat{QHB} = \hat{HBN} = r$; thus $\hat{QBH}$ is an isosceles triangle with $QB = BH$. Further, the triangle QHI is now similar to the triangle BNI, so that $BI/NI = QI/HI$; but since QI is equal to the sum of HB and BI (constant for any position of B on the curve) and thus to the major axis DK,

$$\frac{\sin i}{\sin r} = \frac{QI}{HI} = \frac{DK}{HI} = \text{constant}$$

It is clear that if the refractive indices of the media inside and outside the ellipse have the ratio of the major axis DK to the inter-focal distance HI, *all* rays such as AB incident parallel to the major
axis will reach a common focus in I. The ratio HI/DK is, of course, the eccentricity of the ellipse.

If the second medium has a lower refractive index than the first, the refractive surface must be hyperboloidal (of revolution) (see Fig. 10). A line EBH cutting the surface in B and the focus in H, and the line BI joining B to the other focus are well known to make equal angles \(r\) with the normal CBN which cuts the axis in N. A ray AB parallel to the axis has an angle of incidence \(i = \widehat{ABN}\) which is equal to \(BNH\).

![Diagram](image)

Fig. 10.

If now HQ is drawn through H parallel to NB and cutting BI in Q, it is easy to see that \(\widehat{BQH} = r\); \(\widehat{QHI} = i\), and

\[
\frac{\sin i}{\sin r} = \frac{QI}{HI} = \frac{BI - BQ}{HI}
\]

Also, on similar lines, BH = BQ. If it is remembered that a definition of the hyperbola is that BI minus BH is a constant length equal to DK, the axial interval between the apices of the two branches, we can write

\[
\frac{\sin i}{\sin r} = \frac{DK}{HI}
\]

It is thus possible to assign a fixed refractive index ratio which will make a hyperboloidal surface bring to a common focus all rays incident parallel to the axis and passing from the denser into the rarer medium.

The parallel rays can in either case be regarded as proceeding from an axial object point at infinity. The points of origin and re-union of the rays are called conjugate points.
Descartes discussed also the problem of finding a refracting surface which will unite the rays when both conjugate points are at a finite distance. The principal section of the surface is then a curve of the 4th degree known as a Cartesian Oval. However, such problems are more easily solved by considerations of “optical path” (see Chapter IV).

If a divergent cone of rays has once been made convergent by a suitable refracting surface, and all the rays are converging to a given point, it is possible to pass them again through another spherical surface if the latter is centred in the same point, since all the angles of incidence and refraction will be zero and there can thus be no deviation (Fig. 11). Thus the first surface may involve passing from air into glass, but the second can bring the rays out into air again.

This would be an example of a lens giving perfect ray-union for light of a single frequency, though (owing to the usual change of refractive index of all media on a change of wavelength) the union of rays of different colours will not be at the same point, nor will the union of such other rays generally be perfect.

Lenses

Lenses of suitable shape, as well as mirrors, are thus capable of producing sharp real images of axial object points. It is clearly a matter for careful examination in the following parts of this book to find how far the same property can exist for non-axial points. Experiments with lenses show that more or less sharp images are also found for objects near the axis. The images can be projected on a focusing screen or photographic plate.

The Eye

The optical system of the eye, which contains a number of curved refracting surfaces (Fig. 12), produces on the retina (the sensitive
surface) re-union of rays from object points into corresponding conjugate image points in an extended field of vision. The first reasonably complete description of the action was given by Kepler about A.D. 1604. Referring back to the experiment of the pinhole camera (p. 2), it can now be supposed that the rays reflected or diffused from any illuminated element of the screen, which enter the lens system of the eye, are re-united in a corresponding point on the retina so that we "see" the image. However, we often see "images" which are not first formed on a receiving screen outside the eye; for example the images formed by reflection at a plane surface such as that of still water.

Virtual Images

Let B (Fig. 13) be a point object near a plane reflecting surface, the trace of which is AP, the surface being imagined perpendicular to the page. Drop a perpendicular BA to the surface, and produce to B', making AB' equal to BA. If P is any other point on the surface, join BP and B'P, producing B'P onwards towards Q. Draw also PN, the normal to the surface at P. It is simple to show that PQ is the path of the ray BP after reflection; for BAP and B'AP are two equal right-angled triangles. Since NP is parallel to AB', \( \hat{BPN} = \hat{ABP} \), and \( \hat{NPQ} = \hat{ABP} \). However, \( \hat{ABP} = \hat{ABP} \), so that \( \hat{BPN} = \hat{NPQ} \) wherever P may be. Thus all rays after reflection have paths which, if produced backwards, intersect in B'. Similarly all rays from any other point C appear after reflection to diverge from a unique point C' found by a similar construction.

An eye, observing from above the surface, can receive rays from the points B, C, etc., directly; or can receive the reflected rays, thus
observing $B'C'$, etc. The configuration of the images $B'$, $C'$, etc., is symmetrical (with respect to the plane surface) with the actual object parts. The viewing eye can "see" the sharp images, though the mirror alone cannot project them on a focusing screen. Such images are virtual images as distinct from the real images in which the rays are re-united outside the eye.

Now if a camera (say) forms the images of the virtual images $B'$, $C'$, etc., we can speak of the latter as virtual objects. Moreover, it is possible to give a lens such a shape (Fig. 14) that the rays from

![Fig. 13.](image)

an object-point, after passing through it, appear to diverge accurately from some other point; such a point (which is still regarded as a "conjugate" point) is yet another example of a virtual image. We shall see, later, that reasonably good image-formation is possible with certain lenses for a wide range of object-distances; the image may be real or virtual according to the distance of the object.

Accordingly we may formulate a general scheme as follows (see Fig. 15) for a dioptric system, which we imagine to receive light rays travelling from left to right—

(i) A real object may, of course, be placed at any point to the left of the first surface $A$, of the system.

(ii) If rays of light are converging towards some point to the right of the surface $A$, the object will be virtual.

Thus the "object space" exists through all the space on both sides of $A$, but the real part of the object space is to the left, and the virtual part to the right of that surface.
(iii) A real image may be formed at any point to the right of B, the last surface of the system.

(iv) If rays of light are diverging from some point on the left of B when they leave the system the image will be virtual.

Thus the "image space" also exists through all the space on both

![Diagram](image)

**Fig. 14.** Virtual images produced by

(a) Negative lens,

(b) Positive lens.

![Diagram](image)

**Fig. 15.**

sides of B, but the real part of the image space is to the right and the virtual part to the left of that surface.

**Action of Spherical Refracting Surfaces**

The foregoing paragraphs on mirrors and lenses have indicated the useful optical properties of specially chosen axially symmetrical surfaces such as ellipsoids and the like. If, however, the apertures are symmetrical with the axis and very small, an ellipsoid of revolution departs very little from the spherical surface having the same
radius of curvature at the pole. The ordinary equation* of the ellipse with semi-axis $a$ and $b$ (principal section of the ellipsoid) is, referred to the centre,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With respect to an origin at the vertex, however, it is

$$\frac{(x - a)^2}{a^2} + \frac{y^2}{b^2} = 1$$

(the abscissa being written $x$). Solving this quadratic equation for $x$ we obtain

$$x = a \left\{ 1 \pm \left( 1 - \frac{y^2}{b^2} \right)^{\frac{1}{2}} \right\}$$

Taking the negative sign for the inner bracket as the one clearly relevant to the part of the curve near the origin, and expanding by the binomial theorem,

$$x = \frac{ay^2}{2b^2} + \frac{ay^4}{8b^4} + O(y^6) \quad . . . . \quad (1.09)$$

where the symbolic term $O(y^6)$ means that the remaining terms contain no powers of $y$ of order lower than $y^6$.

The corresponding formula for a circle of radius $r$ is immediately derived from (1.09) by putting $b = a = r$; then if $x_c$ is the abscissa for the circle,

$$x_c = \frac{y^2}{2r} + \frac{y^4}{8r^3} + O(y^6)$$

It should be noted that the section of any continuous axially symmetrical surface may be represented by a similar series of terms in even powers of $y$.

Now if we choose a spherical surface of radius $r$ such that the coefficients of $y^2$ are the same in each series, i.e. $r = b^2/a$, this ensures (in accordance with Newton’s criterion for the radius of curvature at the pole) that the ellipsoid will have the same radius of curvature at the axial point. However, the equation of the spherical section will be equivalent to

$$x_c = \frac{ay^2}{2b^2} + \frac{a^2y^4}{8b^4} + O(y^6) \quad . . . . \quad (1.10)$$

Comparing (1.09) with (1.10) we find that there will be, therefore, a

* N.B. The eccentricity $e$ of the ellipse is given by $e^2 = 1 - b^2/a^2$. 
"horizontal" gap between the two surfaces which will be represented by

\[ x - x_e = \frac{y^4}{8} \left( \frac{a}{b^4} - \frac{a^3}{b^6} \right) + O(y^6) \quad \ldots \quad (1.11) \]

Series such as (1.09) and (1.10) are frequently encountered in optical problems. Let us write a typical expression

\[ x = ax^3 + \beta y^4 + \gamma y^6 + O(y^8) \]

where \( a, \beta, \) and all the other similar coefficients are finite and real. By taking sufficiently small values of \( y, \) it can easily be secured that any higher term becomes small in comparison with the one preceding it. Such a series is convergent if \( y \) can always be chosen so small that the remainder after \( n \) terms is less than any chosen number. For example, the remainder after the terms in \( y^2 \) and \( y^4 \) can be made negligible; if these terms are appreciable, they represent the only part of the series which has to be taken into account. Similarly, by restricting \( y \) to values smaller still, the sum of the whole series can be made to approximate, as nearly as desired, to the term in \( y^2. \) Such statements should be verified by taking trial numerical values.

Similarly, it is easy to see that the slope of the tangent to the ellipsoid in the plane containing the axis departs very little from that of the corresponding sphere, provided that the ordinate of the point of contact is sufficiently small. The trigonometrical tangent of the angle between an element of the section of the surface and the \( y- \) axis is \( dx/dy. \) Differentiating, we find from (1.09) and (1.10) respectively,

\[
\begin{align*}
\frac{dx}{dy} &= ay + ay^3 + O(y^5) \\
\frac{dx_e}{dy} &= \frac{ay}{b^2} + \frac{a^3y^3}{2b^6} + O(y^5).
\end{align*}
\]

\[
\begin{bmatrix}
\frac{dx}{dy} \\
\frac{dx_e}{dy}
\end{bmatrix}
= \begin{bmatrix}
ay + ay^3 + O(y^5) \\
ay/b^2 + a^3y^3/2b^6 + O(y^5)
\end{bmatrix}
\quad \ldots \quad (1.12)
\]

It can be argued in precisely the same way that by making \( y \) sufficiently small the sum of either series may be made to approximate as closely as desired to the value of the term in \( y. \)

The area of the surface for which the slope \( dx/dy \) is sufficiently nearly represented by the term in \( y \) is the "paraxial" region of the surface.

It may be noted that the powers of \( y \) concerned in the successive terms in (1.12) are lower than in the series (1.09) or (1.10), so that in general differences of slope may be expected to be sometimes appreciable even when the actual gap between the two surfaces remains inconsiderable. The slope of the tangent to a refracting surface in a
principal section determines the refracted direction of a given ray incident at the point of contact. Hence it is established that, over a sufficiently small aperture symmetrical with the axis, the refractive effects of an ellipsoidal surface can be imitated as closely as desired by a spherical refracting surface having the same radius of curvature at the pole.*

A similar argument may be applied to the case of any axially symmetrical surface, such as the Cartesian ovaloid, of which the abscissae are represented by a similar series of even powers of \( y \). (Note that the axial symmetry must exclude odd terms in \( y \) because the equality of abscissae of points on the curve represented by numerically equal values of \( y \) must be independent of the sign.) Therefore it may be expected that refracting systems of small aperture can be made to yield sharp images even if they employ spherical surfaces in place of the special ellipsoids, etc., which elementary ray theory indicates. The advantages of the nonspherical forms seem lessened when the facts of dispersion are remembered (see p. 73). It will be shown that in general these effects of dispersion must be countered by the use of combinations of lenses using different glasses, and that in designing these combinations it is also possible to balance out the errors in monochromatic ray-union (aberrations) which the use of spherical surfaces in systems of appreciable aperture would otherwise entail. The monochromatic aberration characteristic of an axial image point is called "spherical aberration."

The use of spherical surfaces has other special attractions. They are relatively easy to produce with the necessary great accuracy of contour, and for this reason: Two spheres or segments of spheres of the same radius (one concave and the convex) will fit each other in all relative positions. Consequently by grinding together the spherical surfaces of an iron tool and a piece of glass with an abrasive such as carborundum, the regions of mutual contact receive the greater wear. Excrences on each are therefore progressively removed, and it is possible by such means to produce spherical refracting surfaces of exceedingly high accuracy. Special books† should, however, be consulted for the complex techniques of grinding and polishing. It has not been found possible, hitherto, to produce aspherical surfaces with comparable facility, and the optics of spherical surfaces is therefore still of major importance.

* It can be shown that a still better approximation between sphere and ellipsoid can be given, if the aperture is finite, by choosing a modified radius of curvature for the sphere.
THE LAWS OF GEOMETRICAL OPTICS

Yet another consideration which has been of much importance in the past is that the numerical calculations employed in the design of optical systems are relatively simple if all the surfaces employed are to be spherical.

For all these reasons, much attention has still to be given to image formation by spherical refracting and reflecting surfaces. However, the elements of the “paraxial” theory to be discussed later on apply to any axially symmetrical systems provided that the usable area of any continuous refracting surface is supposed to be so near the axis that it is indistinguishable in its optical effects from a spherical surface of the same radius at the vertex. The “paraxial region” of a complete system can be imagined as a tube-like space around the axis, enclosing all such areas of the refracting surfaces; and “paraxial rays” are those rays (not necessarily rays in axial planes) which lie within such regions. All paraxial rays make very small angles with each other both inside and outside the system.

Of course, the distinction between the paraxial regions and those exterior thereto is a matter of approximation. The values which will be discussed are those which would be found if the apertures became smaller and smaller, but optical systems and ray re-union need not be absolutely perfect in order to obtain satisfactory results. Owing, in part, to the finite wavelength of light there is a considerable tolerance.* Paraxial theory has, therefore, a close relevance to practical cases. Moreover, it formulates ideas which are relevant in the optics of “corrected” systems.

The Optics of a Spherical Refracting Surface (Monochromatic Light)

It will be convenient first to discuss the refraction of monochromatic rays lying in an axial plane of a spherical refracting surface which separates two homogeneous media of refractive indices n and n' respectively; see Fig. 16, where AC is the axis and C is the centre of curvature. A ray such as DP in the figure must be determined by two parameters; for example by the angle between the ray and the axis, and the distance from the pole at which (perhaps after being produced) it intersects the axis (shown respectively as U and AB). As in any problem in analytical geometry, a sign convention is required in order to distinguish rays which are diverging from or converging towards the axis, or between points of intersection which lie to the left or to the right of the origin of reference. There is no loss of physical generality in supposing, for problems of optics, that the initial direction of the light is always from left to right since an instrument can always, in theory, be reversed with respect to the

* This point is discussed below; see p. 198.
user, but a possible reversal of direction of the light inside the instrument, as in the case of reflection, must be provided for.

The positive and negative directions from the origin are taken as in the usual Cartesian system (positive to the right and upwards, negative to the left and downwards). The usual sign convention for angles is, however, one which does not agree with the one usual in other analytical work, but it is found convenient for optical problems and has been used in many of the most important continental works on the subject. The acute angle between a ray and the axis is positive when a clockwise turn will bring a ruler from the axis direction to that of the ray, and vice versa. The radius of curvature is positive when the centre lies to the right of the pole, and vice versa. The parameters in Fig. 16 are all positive.

It is simple to calculate the data for the refracted ray from those of the incident. The angle of incidence is $\hat{CPB} = I$, the angle of convergence $\hat{CPB} = U$. Let the radius of curvature $AC$ be $r$ and the intersection distance $AB$ be $L$.

The angle of incidence is given by elementary trigonometry. From the triangle CPB,

$$ \sin I = \frac{(L - r) \sin U}{r} \quad . \quad (1.13) $$

The refracted ray intersects the axis in some point $B'$. The angle of

* The sign convention for the acute angle of incidence is not often needed in practical calculations, but is opposite, i.e. it is positive if an anti-clockwise turn will bring a ruler from the direction of the normal to the direction of the ray; similarly for the refracted ray.
refraction is $I'$, and the angle of convergence is $U'$. The exterior angle $\angle ACP = \theta$, and is equal to the sum of the interior angles:

$$\theta = U + I = U' + I' \quad \ldots \quad (1.14)$$

Hence

$$U' = U + I - I' \quad \ldots \quad (1.15)$$

The angle of refraction is found from

$$\sin I' = \left(\frac{n}{n'}\right) \sin I \quad \ldots \quad (1.16)$$

the value of $I'$ being found from a table of sines; it is then possible to calculate $U'$, one of the parameters of the refracted ray, from (1.15). The other parameter $L'$ is then found from the equation corresponding to (1.13) above, i.e.

$$L' - r = \frac{r \sin I'}{\sin U'} \quad \ldots \quad (1.17)$$

While the foregoing procedure, represented by equations (1.13) to (1.17), represents a convenient scheme for numerical calculation, it provides no convenient analytical formula for the data of the refracted ray; it is only possible to give simple expressions for the paraxial case.

The law of refraction gives (see (1.14) above)

$$n \sin (\theta - U) = n' \sin (\theta - U')$$

in which the values of $(\theta - U)$ and $(\theta - U')$ are respectively the angles of incidence and refraction. Now the usual expansion for the sine of an angle can be written

$$\sin I = I - \frac{I^3}{3!} + \frac{I^5}{5!} - O(I^7)$$

we see that the replacement of a sine by the corresponding angle (radian measure) would involve only the neglect of quantities of the order of the cube of the angle. The concept of the paraxial region of the surface was discussed above.

If the radius $r$ is finite while $y$ (the height of incidence) is so relatively small as to be in the paraxial region, then $y/r$ is a small but appreciable quantity, but $(y/r)^3$ will be negligible; however, when $y$ is very small, $y/r$ is indistinguishable from $\theta$. Also $U$ is very small in the same sense if the ray is paraxial; its cube will be negligible. Hence these assumptions about the paraxial rays make it sure that $(\theta - U)^3$ and $(\theta - U')^3$ will both be inappreciable; and thus for the paraxial region we may simplify the law of refraction by writing the angular measure in place of the sine. It is advisable to indicate the

2—(T.781)
quantities of the paraxial region by a special notation, in this instance by the use of small italic letters in place of the corresponding capitals.

The relation then becomes

\[ n(\theta - u) = n'(\theta - u') \]

or

\[ n'u' - nu = y(n' - n)/r \]  \hspace{1cm} (1.18)

In this case the incident and the refracted ray are completely determined by this common point of intersection in the surface and their angles with the axis before and after refraction. If, however, we divide the equation throughout by \( y \), we obtain

\[ n'\left(\frac{u'}{y}\right) - n\left(\frac{u}{y}\right) = \frac{n' - n}{r} \]

Note, however, that in the paraxial limitation, the values of \( u' \) and \( u \) are indistinguishable from their trigonometrical tangents.*

Then if \( l \) and \( l' \) are the intersection distances of the paraxial ray with the axis, before and after refraction, measured from the pole of the surface, we have (with increasing accuracy as \( y \) gets smaller and smaller)

\[ \frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{r} \]  \hspace{1cm} (1.19)

The quantity \((n' - n)/r\) is called the power of the surface and will be denoted by the capital letter \( F \).

\[ F = \frac{n' - n}{r} \]  \hspace{1cm} (1.20)

The values \( n/l \) and \( n'/l' \) are the vergencies of the ray before and after refraction.† Both of these, and the power, are given in diopters if the lengths are expressed in metres.

The great importance of equation (1.19) is that the conjugate distance \( l' \) resulting from a given object distance \( l \) can be taken as independent of the intersection height for all paraxial rays, and the image formation is thus indistinguishable from a perfect ray re-union for the paraxial region. This argument remains valid for all continuous axially symmetrical surfaces whether they are spherical or not.

* N.B. \( \tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \text{etc.} \)

† If the equation is written in the form \( n'/l' = n/l + (n' - n)/r \), it can be remembered by the mnemonic: final vergence = initial vergence + power.
The Focal Lengths

If the object distance \( l \) is infinite, the image distance \( l' \) (see equation (1.19)) is denoted by \( f' \), where

\[
f' = \frac{n' r}{n' - n} . \quad \quad \quad (1.21)
\]

and this is the “focal length of the image space” in this particular case. Correspondingly, if the object distance is such as to make \( l' \) infinite, then \( l \) in this case is the focal length of the object space, denoted by \( f \), where

\[
f = -\frac{n r}{n' - n} . \quad \quad \quad (1.22)
\]

From equation (1.20), the relation between these focal lengths and the power \( F \) of the surface is seen to be

\[
F = -\frac{n}{f} = \frac{n'}{f'} \quad \quad \quad (1.23)
\]

The formula obtained above for refraction at a single spherical surface will be unchanged and valid for cases in which the signs of the parameters may be positive or negative. In using such equations the correct sign must be attached to the numerical value on substitution for the symbol.

The Thin Lens

Let the radii of curvature of the surfaces of a thin lens of refractive index \( N \) in air be \( r_1 \) and \( r_2 \) respectively in the order of their encounter by the light, and let the object point be at a distance of \( l \) from the first surface, on the axis of symmetry. The refraction at the first surface is represented by

\[
\frac{N}{l'} - \frac{1}{l} = \frac{N - 1}{r_1} . \quad \quad \quad (1.24)
\]

If the axial thickness of the lens is \( d \), the distance of this first image from the second surface is \( l' - d \); it becomes the “object distance” for the second surface, and the refraction is thus represented by the equation

\[
\frac{1}{l'_2} - \frac{N}{l'_1 - d} = \frac{1 - N}{r_2} . \quad \quad \quad (1.25)
\]

where \( l'_2 \) is the distance of the final axial image from the second surface. The two foregoing equations can be used for numerical
calculations if \( d \) is appreciable. However, if \( d \) is negligible in comparison with \( l_1' \) it can be omitted from the last equation, and the addition of (1.24) and 1.25) then gives

\[
\frac{1}{l_2'} - \frac{1}{l_1} \simeq (N - 1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = (N - 1)R .
\]  

(1.26)

where \( R \) is written for \( (1/r_1 - 1/r_2) \) and is called the "total curvature." The quantity on the right-hand side of the equation is known as the power \( F \).

\[
F = (N - 1)R .
\]  

(1.27)

Its significance can be more fully appreciated by multiplying (1.26) throughout by \( y \), the height of incidence of some paraxial ray passing between a pair of conjugate points on opposite sides of the lens. The value of \( y/l_1 \) then becomes \( u_1 \) in this paraxial case, i.e. the angle between the initial ray and the axis. The supposed thinness of the lens* makes any difference of height in the two surfaces negligible, so that (1.26) now becomes

\[
u_2' - u_1 = yF .
\]  

(1.28)

which means that the deviation of such a ray in an axial plane is proportional to the height of incidence; it does not depend on the conjugate distances. Exactly as in the case above, we can find the image point distance \( f' \) which results from an infinite object distance; it is the focal length of the image space

\[
f' = \frac{1}{F} .
\]  

(1.29)

and the focal length \( f \) of the object space is, correspondingly,

\[
f = -\frac{1}{F} .
\]  

(1.30)

As before, if the metre is used as the unit of length the power will be given in diopters. The vergencies in this case are simply the reciprocals of the conjugate distances (the correct sign being always inserted in numerical examples) and the same mnemonic can be used. The image space focal length \( f' \) of a thin lens is often useful in a conjugate distance calculation, since

\[
\frac{1}{l_2'} - \frac{1}{l_1} = \frac{1}{f'} = F .
\]  

(1.31)

* The concept of a "thin lens" is, of course, a mathematical abstraction, since, if the axial thickness vanishes, the difference of curvature of the two surfaces on which the power depends would make the aperture of a lens of positive power necessarily zero. However, it is possible to obtain approximate relations which are useful in practice when the thickness, though appreciable, is still very small in comparison with the radii.
but the term focal length has a much richer significance than might appear from the foregoing discussions of conjugate distances, as will appear below. Fig. 17 represents the relation between the conjugate distances for a lens of power $+5.0$ D (plus five diopters); focal length $= 20$ cm.

![Graph showing conjugate distance relations for a + 5.0 D lens (distances in metres).]

**Object and Image Fields; Single Spherical Refracting Surface**

Let a spherical refracting surface AP (Fig. 18) separate media of refractive indices $n$ and $n'$. Let B and B' be paraxially conjugate points. It can be imagined that the limitation of the rays to the paraxial region is secured by a small symmetrical circular aperture in a diaphragm at C, the centre of curvature of the surface, in the medium $n'$.

Now if BB$_1$ and B'B$_1'$ represents the trace of spherical surfaces, also centred in C, and B$_1$CB$_1'$ is also an axis of symmetry of the refracting surface, it follows that associated paraxial rays may surround it, and that B$_1'$ and B$_1$ will also be conjugate points. If the object field is a plane surface, with trace BB$_0$, perpendicular to the axis of symmetry BAC, and B$_0$ is on the line B$_1$CB$_1'$, it will be under-
stood that the transfer of the object point from $B_1$ back to $B_0$ (Fig. 18) will entail a corresponding movement of the conjugate point $B_1'$ back to $B_0'$. The equation (1.19) when differentiated with respect to $l$, gives—

$$\frac{dl'}{dl} = \frac{n}{n'} \frac{l'^2}{l^2}$$

(1.32)

so that small shifts of the object point entail a shift (having the same sign) of the image point. The image field will therefore now be more heavily curved than the sphere $B_1'B'$, and a plane object field will have a curved image field. If object points in such a field as $BB_0$ were self-luminous, and a plane screen perpendicular to the axis were placed at $B'$ there would evidently be good definition of the point images only over so small an area of the surface for which the gap between the plane screen and the surface of sharp images is optically negligible.* Owing again to the finite wavelength of light, good definition is, in fact, obtained over an appreciable area.

The image field may be treated as plane within the area outside which the gap between the plane and the curved surface becomes appreciable. If the vertex radius of the image field is $r_i$ and distance from the axis is denoted by $h$, the gap is given sufficiently nearly by the "spherometer formula"

$$\text{gap} \approx \frac{h^2}{2r_i}$$

and we shall therefore confine the discussion for the present to cases of axially symmetrical object and image fields which are so small that while $h$ is appreciable, $h^2/2r_i$ is negligible. It is clearly an important task for later theory† to examine how far it may be possible to produce large plane images suitable for a projection screen or for photography on plane films or plates.

* The criterion is discussed below; see p. 198, and Appendix III.
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It should now be noted carefully that within the limits defined in the last paragraph we may discuss plane conjugate surfaces perpendicular to the axis for a single spherical refracting surface. Few of the rays from object points very near the axis (extra-axial points) are confined to axial planes, but they can still be paraxial rays which may be pictured within the narrow tube-like space surrounding the main axis of symmetry. If a second refracting surface is added to the system the image surface for the first refraction becomes the object surface for the second, and so on. (Some surfaces might in practice enhance, and some might diminish the actual physical curvature of the succeeding images but this is immaterial for the paraxial discussion.)

The subject matter of the paraxial theory, as developed in the analytical sense chiefly by Gauss, presumes that the refraction of a ray in an axial plane always takes place in accordance with an equation like (1.18), and that the conjugate surfaces are always plane and perpendicular to the axis. There is, however, one difficulty which should first be examined. The discussion of the case pictured in Fig. 18 assumed that the "stop" or aperture was placed at C, the centre of curvature of the surface. The addition of a further surface would, in general, then mean that the principal ray of the incident pencil which has passed through the centre of the stop associated with the first surface would not intersect the centre of curvature of the second. This entails a loss of symmetry in the refraction of the pencil, and it is important to examine the effect on the refraction.

Tangential and Sagittal Rays

Fig. 19 represents three-dimensional aspects of Fig. 18 above, but no paraxial restrictions are in question for the moment. The axis of the spherical refracting surface is BACB', and PAQ represents the section of the surface by the plane containing the axis and the extra-axial object point B₀. The curve RAS represents a section of the surface by the axial plane perpendicular to the first. The approximately circular line FSQR represents as it were a "parallel of latitude" on the sphere with respect to the pole A (or a section by a plane perpendicular to the axis, not far from the pole). Imagining now that rays from B₀ proceed to P, Q, R, and S, as well as to A, it is possible to picture a group of four rays directed to the edge of an aperture coincident with PSQR, and surrounding the principal ray B₀A, not shown in the figure. Now, the rays B₀R and B₀S have symmetry with respect to the axial plane containing B₀, and thus with respect to B₀A; but there is no such symmetry in the case of the rays B₀P and B₀Q with respect to B₀A, though these three rays
lie all in the one axial plane containing \( B_0 \), so that their paths after refraction can be discussed by plane trigonometry (see below).

*If the aperture is small, \( B_0R \) and \( B_0S \) are called *sagittal* rays; and \( B_0P, B_0Q \) are *tangential* rays with respect to the principal ray \( B_0A \). All the close pairs of rays with sagittal symmetry constitute a “sagittal fan,” and the term “tangential” fan is used in a similar way.

The first problem is now to discuss the refraction of the sagittal rays. The normal to the surface at \( S \) can be drawn by joining \( C \) and \( S \). Now if we join \( B_0C \) we obtain a new axis of symmetry of the

![Fig. 19.](image)

refracting surface, so that the plane containing \( B_0C \) and \( CS \) is the plane of incidence of the ray \( B_0S \); the refracted ray will lie in this plane and will also intersect the new axis at some point which can again be calculated by trigonometry. Owing to the symmetry of the points \( R \) and \( S \) with respect to the axial plane through \( B_0 \), the ray refracted at \( R \) must cut the new axis, \( B_0C \) produced, in the same point, which we will call \( B'_s \) (the image point of the sagittal rays). The problem is represented in Fig. 20, where the rays in the plane containing the normal \( SC \) are shown. It is useful to find an expression connecting \( SB_0 \) and \( SB'_s \) (written \( b \) and \( b'_s \) respectively) in terms of the relevant parameters, including in this case the angles of incidence and refraction (written \( \alpha \) and \( \alpha' \) respectively). In order to obtain the quantitative relations all the magnitudes in Fig. 20 will be treated temporarily as positive, and the relation of the resulting formula to the general strict sign convention will be discussed later.
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Now the area of the triangle $B_0Sb'_s$ is given by

$$\Delta B_0SB'_s = \Delta B_0SC + \Delta SCB'_s$$

(using the symbol $\Delta$ here to indicate an area), then

$$\frac{1}{2}B_0S \cdot SB'_s \sin(\alpha - \alpha') = \frac{1}{2}B_0S \cdot SC \sin \alpha + \frac{1}{2}SC \cdot SC \sin \alpha'$$

Thus

$$bb'_s \left( \sin \alpha \cos \alpha' - \cos \alpha \sin \alpha' \right) = br \sin \alpha + b'_s r \sin \alpha'$$

On multiplying throughout by $nn'$, and dividing by $n \sin \alpha$ or its equivalent, $n' \sin \alpha'$,

$$bb'_s \left( n' \cos \alpha' - n \cos \alpha \right) = n'br + nb'_s r$$

![Fig. 20.](image)

or, dividing by $bb'_s r$ and rearranging,

$$\frac{n'}{b'_s} + \frac{n}{b} = \frac{n' \cos \alpha' - n \cos \alpha}{r} \quad . \quad (1.33)$$

a formula which expresses a relation between the magnitudes shown in Fig. 20, when all these are assumed to be numerically positive. This expression holds, of course, for the rays $B_0S$ and $B_0R$ (Fig. 19) at any aperture, though indeed the problem of calculating $\alpha$ and $\alpha'$ has not been examined. However, as the sagittal aperture $RS$ becomes very narrow, the angle of incidence of both rays approximates the more closely to that of the central ray $B_0A$, and $\sin \alpha$ will then be known; i.e. $BB_0/B_0A = h/s$ where $h$ is written for the object height $BB_0$, and $s$ for the distance $B_0A$ of the sagittal object point, measured along the principal ray from the point of refraction. Also the point of re-union of the rays of a very narrow sagittal “fan” of rays is closer and closer to the refracted principal ray from $A$ as the aperture diminishes, so that its limiting position is taken as on that ray.

The corresponding calculation for tangential rays only concerns rays in one plane; it is required to find the limiting relations when the ray divergence is similarly small. In Fig. 21 the points $P$ and $Q$ of incidence of two tangential rays from $B_0$ are shown as close to $A$. 
The perpendicular QT is dropped from Q to B₀P; then, as Q and P approach each other at A, the figure TQP becomes increasingly like a small triangle in which the angle \( \widehat{PQT} \) is equal to the angle of incidence \( \widehat{BAB₀} \) of the mean ray \( B₀A \), which is written as \( \alpha \).

Thus in the limit

\[
B₀A(PB₀Q) = PQ \cos \alpha
\]

In a similar way, taking \( B_t' \) as the crossing point of the rays from P and Q after refraction

\[
AB_t'(PB_t'Q) = PQ \cos \alpha'
\]

![Fig. 21.](image)

(imagine \( AB_t' \) to be the limiting position of both the rays when P and Q approach each other, the angle of refraction being \( \alpha' \)).

Now in altering the point of incidence from Q to P, the incident ray moves anti-clockwise; this alone would tend to produce a numerical increase of the angle of incidence in our case; but the normal moves clockwise into the bargain. Hence the total increase \( \delta \alpha \) in the numerical angle of incidence is the sum of \( \widehat{PB₀Q} \) and \( \widehat{FCQ} \), i.e., when both these angles are very small,

\[
\delta \alpha = \frac{PQ \cos \alpha}{B₀A} + \frac{PQ}{r}
\]

However, while the angle of refraction would be increased by the clockwise movement of the normal, it is diminished by the lesser clockwise movement of the ray so that in the limit when PQ is very small

\[
\delta \alpha' = \frac{PQ}{r} - \frac{PQ \cos \alpha'}{AB_t'}
\]
The law of refraction when differentiated gives (if \( \alpha \) is the angle of incidence)

\[ n \cos \alpha \, d\alpha = n' \cos \alpha' \, d\alpha' \]

Thus

\[ n \cos \alpha \left( \frac{PQ \cos \alpha}{B_oA} + \frac{PQ}{r} \right) = n' \cos \alpha' \left( \frac{PQ}{r} - \frac{PQ \cos \alpha'}{AB_t'} \right) \]

Dividing throughout by PQ, the equation becomes, after rearrangement,

\[ \frac{n' \cos^2 \alpha'}{AB_t'} + \frac{n \cos^2 \alpha}{B_oA} = \frac{n' \cos \alpha' - n \cos \alpha}{r} \quad . \quad (1.34) \]

As before, it will be clear that as PQ becomes shorter and shorter it tends to lose significance in the refraction equation. Therefore, the whole fan of tangential rays associated with \( B_oA \) tends to come to a focus at a point \( B_t' \) which is taken to be "on" the principal ray.

**Sagittal and Tangential Formulæ Adhering to the Sign Convention**

The standard sign convention set out above (p. 24) can be used for lengths measured along the principal ray if the sign given to any such length is that of its projection on the axis. Thus in Fig. 20 the distance of \( B_o \) from the point of incidence \( S \) would be numerically negative, while the distance \( SB_t' \) will be positive. The use of new symbols \( s \) and \( s' \) respectively for these distances will indicate the adherence to the sign convention. The angles of incidence were denoted by \( I \) and \( I' \) in the trigonometrical scheme, and this notation will now be resumed; but since only the cosines appear in the trigonometrical formula no sign changes are involved here. Similarly, \( t \) and \( t' \) signify the tangential conjugate distances. The two formulæ, (1.33) and (1.34), now take the form—

**Sagittal fan:**

\[ \frac{n'}{s'} - \frac{n}{s} = \frac{n' \cos I' - n \cos I}{r} \quad (1.35) \]

**Tangential fan:**

\[ \frac{n' \cos^2 I'}{t'} - \frac{n \cos^2 I}{t} = \frac{n' \cos I' - n \cos I}{r} \quad (1.36) \]

They may be compared with the formula for the axial conjugate distances—

\[ \frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{r} \]

and it is seen that both the sagittal and tangential forms reduce to the axial form when the angle of incidence is so small that the cosine approximates to unity. Of course, the sagittal and tangential fans of rays are only particular cases. Referring to Fig. 19, it will be
clear that we might consider rays in meridians between PQ and RS. However, it can be shown that, while the aperture of such fans is not larger than a first-order small quantity, their foci will also be indistinguishably close to the principal ray and will fall within the positions of the tangential and sagittal foci. Note carefully that while the aperture of the fans is limited, the formulae impose no restriction on the values of the angles of incidence; it is thus possible to discuss the image formation for objects at any distance from the axis.

Let the object points be situated, for example, in a plane field at an infinite distance, and let the aperture be coincident with the surface (Fig. 22). Then

\[ l' = \frac{n'r}{n' - n}, \quad s' = \frac{n'r}{n' \cos I' - n \cos I} \quad \text{and} \quad t' = \frac{n'r \cos^2 I'}{n' \cos I' - n \cos I} \]

If the surface is of positive power, it will be found that the sagittal and tangential focusing distances are progressively shorter than the axial distance. Moreover, since the values of \( s \) and \( t \) are independent of the signs of \( I \) and \( I' \), the "image fields," i.e. the respective foci of the tangential and sagittal focusing points, will be symmetrical with the axis, and both concave towards the pole of the refracting surface. A similar condition can be inferred when the object field is perpendicular to the axis and at a finite distance; further, provided that the object field is a continuous surface and symmetrical with the axis, both the image fields will be continuous and symmetrical.

In the discussion on p. 29 it was shown that, if the stop were situated at the centre of curvature of the surface, all the rays sufficiently closely associated with a principal ray would tend to reunite in one point. Although the image field would generally be
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curved and symmetrical with the axis (provided the object field were symmetrical) there would then be no difference for the tangential and sagittal rays. Thus the appearance of this difference, which is known as "astigmatism of the oblique pencils," is associated with the axial shift of the stop, but the symmetry of the fields with regard to the axis is unchanged. It may now be argued that if the heights $h$ of objects and images are always so small with respect to the radius of curvature $r_i$ of such a field that $h^2/2r_i$ is negligible, the gaps between either of the sagittal and tangential curved fields and the plane surface touching the image fields at their vertices will be inappreciable. The difference between the "$s$" and "$t$" foci is then also negligible; hence, assuming paraxial limitations both as to object height and the aperture of the refracting surface, and diaphragm, the terms "object plane" and "image plane" still have a significance independent of any consideration of stop position. If this applies to a system of one refracting surface, it must (with the proper ray limitations) be valid for a system of any number of surfaces. Anticipating a point which will be more fully discussed in Chapter VII (p. 184), the totality of the rays which can enter an aperture from an off-axis object point as in Fig. 19 could be divided up into sets in chords parallel to RS, and each set could be associated with a principal ray incident where the chord intersects PQ. We have established that all these principal rays (which form a tangential fan) will (if the aperture is small) meet in a point; but after they have met, the associated sets of sagittal rays will meet, each set on its own principal ray. The result is that all the rays entering the aperture pass through two "focal lines"; one in a radial direction in the field (where the sagittal rays meet) and one in a tangential direction (tangential to a circle round the axis) where the tangential rays meet, but where the sagittal fans still have a finite width. The "lines" have an appreciable size only when the gap between the "$s$" and "$t$" fields is appreciable. Thus it can be argued that in so far as the proper restrictions are imposed with respect to the distances of object points from the axis and also with regard to the allowable apertures of the surface, all the rays from a point in an object plane perpendicular to the axis in the object field will tend to re-unite in a conjugate point in an image plane also perpendicular to the axis. The discussion has covered rays not lying in an axial plane.

Image Formation by a Thin Lens

The previous discussion was limited to the derivation of the formulae for the distances of axial conjugate points. However, it
is now shown that, if paraxial limitations hold, planes containing these axial points and perpendicular to the axis will contain pairs of extra-axial conjugate points of sharp ray re-union. If then we can trace even a single ray through a lens from one plane to its conjugate plane, we shall locate the sharp unique image point.

No matter what the shape of a thin lens, the axial symmetry of its two surfaces will ensure that the elements of the surfaces crossed by a paraxial ray incident at the pole of one of them are so nearly parallel that the ray will not be appreciably deviated in direction,

and will suffer no appreciable lateral displacement if the lens is indeed of negligible thickness. This accords with the deviation formula (1.28)

\[ u' - u = y F' \]

for if \( y \) is zero the ray is undeviated. Hence we can, in geometrical constructions, draw one ray \( B_1 A \) (Fig. 23 (a) ) through the centre of the lens. If the axial conjugate points \( B \) and \( B' \) are known (or are calculable) the image point \( B_1' \) can thus be ascertained. We must, however, examine the conditions under which diagrams like Fig. 23 may properly be constructed and used. The validity of the arguments about ray re-union is dependent on maintaining the extreme smallness of object heights and apertures. We can, however, adopt the expedient of using a vertical scale highly magnified in comparison with the horizontal. If all vertical heights are supposed to be increased in the ratio of, say, 10 to 1, the clarity of the diagrams can be retained and the numerical results of the construction can be
improved in accuracy. Most of the diagrams in Chapter II are to be interpreted in this way. The thin lens is sometimes shown in such diagrams conventionally as a straight line with a V at the top, inverted or erect accordingly as the power is positive or negative (see Fig. 23 (b)).

Sometimes the data given may be the distance of the object plane, the height $h$ of the object $BB_1$, and the focal length $AF'$. The position of the second conjugate plane and the height of the image may then be ascertained by tracing yet another ray through the lens. The ray from $B_1$ parallel to the axis may be attributed to an axial point of origin at an infinite distance, of which the conjugate point is $F'$, the second principal focus. The ray therefore passes through $F'$ from its point of intersection $P$ in the lens and intersects the first ray in $B_1'$. Since the image point is unique it is completely determined by the intersection of these two rays, and it further determines the unique conjugate plane. If the height of $B_1'$ from the axis is $h'$ (numerically negative on the Figure by the standard convention) then in this special case

$$\frac{h'}{h} = \frac{l'}{l} \quad \quad (1.37)$$

Owing to the possibility of giving the "object distance"* any real value from $-\infty$ to $+\infty$, and remembering the reversibility of the ray paths, consideration will show that the "magnification" defined by $h'/h$ can likewise have any real value; for if there is a real object at infinity on the left, the image has comparatively negligible size and the magnification approximates to zero. If the object approaches from the left the image (inverted) is found to grow. Thus the magnification is negative but will increase numerically. When the object distance (negative) is numerically twice the focal length, the image distance is positive and of the same numerical magnitude (see Fig. 17) so that the magnification is minus one. When the object approaches the first focal point on the left of the lens the image has receded to infinity, and the magnification has accordingly increased to minus infinity. If the distance of the object from the lens is still further decreased the image becomes virtual and erect; the magnification decreases from plus infinity till it becomes plus one, when object and image planes coincide in the thin lens itself. The further range of magnifications correspond to virtual object distances.

An extension of such arguments, or actual experiment, will demonstrate that the attainment of any real value of magnification is possible with any axially symmetrical system of refracting surfaces

* (including virtual object distances).
under paraxial limitations, and that, in the ideal sense, there is only one conjugate plane and only one value of the magnification for any given object or image plane; though of course in practical experiments, owing to the finite wavelength of light, the fallibility of observation, and the small apertures necessary to approximate towards paraxial conditions there may often be a great “depth of focus” within which no appreciable change in definition* of an image can be seen on a screen, even though its size may vary appreciably with the movement of the focusing screen.

The Smith–Helmholtz Relation

An important relation exists in the paraxial case between the heights of object and image and the angles between pairs of rays intersecting in the relevant conjugate planes. Consider the case of a single refracting surface AP (Fig. 24) for which BB₁ and B'B₁' are conjugate planes. Using the standard notation and sign convention, and assuming paraxial conditions,

$$AP = y = lu = l'u'$$  \hspace{1cm} (1.38)

Consider also a ray B₁A refracted at A. Its angle with the axis is \(-h/l\); after refraction it becomes \(-h'/l'\) so that the paraxial form of the law of refraction gives

$$\frac{nh}{l} = \frac{n'h'}{l'}$$  \hspace{1cm} (1.39)

Combining this equation with (1.38) above we obtain

$$nhu = n'h'u'$$  \hspace{1cm} (1.40)

an expression which relates \(h\) and \(h'\) with the angles between the ray BPB' and the axis. However, it will be seen that, if the figure is modified by joining B₁P and PB₁', these intervals must be the two parts of a ray path since B₁' is the unique image point for B₁, and if \(h\) and AP are very small \(\widehat{A}B₁P = \widehat{A}P\), and \(\widehat{P}B₁A = \widehat{P}B₁'A\) with

* See p. 198.
negligible error. Hence, if \( \hat{A} \hat{B} \hat{P} \) and \( \hat{P} \hat{B} \hat{A} \) are written \( \beta \) and \( \beta' \) respectively,

\[
nh \beta = n' h' \beta'
\]

so that the initial and final angles between any pair of paraxial rays intersecting successively in the object and image have the above reciprocal connexion with the object and image heights. This is a most important relation, because it becomes an invariant for any number of successive refracting surfaces if the image for the first surface becomes the object for the second, and so on, i.e. when

\[
n_1 h_1 \beta_1 = n_1' h_1' \beta_1' = n_2 h_2 \beta_2 = n_2' h_2' \beta_2' = \text{etc.}
\]

Thus for any number \( k \) of refracting surfaces,

\[
n_1 h_1 \beta_1 = n_k h_k \beta_k'
\]

In order to find the magnification produced by an optical instrument it is sometimes convenient to trace a paraxial ray, originating from the axial object point, at an angle \( u_1 \) with the axis, through all the surfaces; then, if \( u_k' \) is finally ascertained,

\[
\frac{h_k'}{h_1} = \frac{n_1 u_1}{n_k u_k'}
\]

It is implicit in the paraxial form of the law of refraction, and indeed of the whole concept of this section, that the magnification is single-valued over the paraxial range of object height. Therefore the geometrical configuration of image points in the image plane will be precisely similar to those of the object plane, i.e. the image will be free from appreciable distortion within the paraxial domain, in the sense that the scale of magnification will be uniform.

It has not been shown, of course, that this last property will be retained outside the paraxial region; in fact, distortion is generally present to some extent if the fields are widened (see p. 185 below).

The relations discussed in the foregoing section seem to have been discovered independently by numerous writers, and they have been stated in a number of forms. Huygens (about 1690); the English optical writer C. Smith (A Complete System of Optics, 1738); von Helmholtz (about 1860), and others may be mentioned. The equation has been attributed by some writers to Lagrange.

When the object distance is very great in comparison with the dimensions of the system, the angle \( \beta_1 \) (equation (1.41)) is very small and approximates to \( y_1/l_1 \), where \( y_1 \) is the incidence height in the first refracting surface. Then

\[
n_1 h_1 \beta_1 \rightarrow n_1 h_1 y_1/l_1 = - n_1 y_1 \theta_1
\]
where $\theta_1$ is the (small) angle subtended by the object at the first surface. Hence the Smith–Helmholtz relation now takes the special form—

$$- n_1 y_1 \theta_1 = n_k' h_k' \beta_k' \quad \cdots \cdots (1.43)$$

There is a similar relation for the case when the image distance is relatively very great; it should be written by the student.

**EXERCISES I**

1. A rough measurement of the angular diameter of the sun was made from the earth’s surface immediately before a total eclipse, the result being $0.00944 \pm 1 \times 10^{-5}$ radians. The belt of totality was found to be 1 mile wide, and the approximate distance of the centre of the moon from the point of observation was known to be 240,000 miles. Make the best possible estimate of the diameter of the moon. The distance of the sun can be assumed to exceed $93 \times 10^8$ miles.

2. A man of height 6 ft sees his image reflected in a plane vertical mirror when he is standing upright. Find the minimum height of the mirror required in order that he can see the image at full length, and show that the result is independent of the distance of the mirror.

3. A man, whose eyes are at a distance $h$ above a horizontal floor when he is standing upright, is at a distance $d$ from a vertical wall against which a rectangular plane mirror rests with its lower edge on the ground. The mirror is tilted forward (its base remaining against the wall) until he can no longer see the image of his feet, which were visible at first. The angle of tilt being $\theta$, give an expression for the length of the mirror measured perpendicular to the lower edge.

4. A point source of light is situated in the centre of the base of a short hollow cylinder with inner reflecting walls. The principal section is circular. A screen is held, perpendicular to the axis, at a distance above the cylinder equal to its (uniform) height. Discuss the illumination of the screen.

5. Show that the minimum deviation of a ray refracted in the principal section of a prism of small angle is approximately $(n - 1) A$, where $n$ is the refractive index and $A$ is the prism angle. Taking the numerical case when $A$ is $10^\circ$ and $n$ is 1.60, calculate the error of the formula, and the change of deviation when the angle of incidence on the first face increases by $1^\circ$ from the value for minimum deviation. (N.B. Seven-figure log tables are required.)

6. A solid glass globe of diameter 4 in. has a refractive index of 1.56, and is totally immersed in water of refractive index 1.33. If bundles of parallel rays enter the globe in various directions, find the radius of the surface which is the locus of the final paraxial focus.

7. A sphere of glass of refractive index $n$ encloses a smaller opaque concentric sphere. Show that, as seen from any viewpoint, the apparent angular subtense of the image of the opaque body has a value $n$ times greater than it would be if seen directly, and state the limiting condition under which this can be true.

8. Explain the occurrence of astigmatism through the refraction of a narrow pencil of rays at a plain surface, and give a geometrical derivation of the relevant conjugate distance formulae, i.e. with the usual notation,
Tangential rays: \[ \frac{n' \cos^2 i'}{i'} = \frac{n \cos^2 i}{i} \]

Sagittal rays: \[ \frac{n'}{s'} = \frac{n}{s} \]

9. A bundle of rays enclosing a small solid angle \(d\omega\) is refracted at a surface separating media of refractive indices \(n\) and \(n'\) respectively. Assuming the results of exercise 8 above, show that if \(i\) and \(i'\) are the angles of incidence and refraction, and \(d\omega'\) is the solid angle enclosed by the refracted rays,

\[ n' \cos^2 i' \ d\omega' = n \cos^2 i \ d\omega \]

10. An ellipsoid of revolution has an eccentricity of 0.94. The radius of curvature at the pole is 5.0 cm. A spherical surface of the same radius is tangent at the pole. Calculate the height from the common axis of symmetry within which the gap between the two surfaces is less than the wavelength of light, i.e. \(0.5 \times 10^{-4}\) cm. (N.B. The sixth-power term in the expression for the gap may be neglected.)

11. With reference to the two surfaces in example 10, find the height from the axis within which the tangents to the principal sections (drawn at the same height) are parallel to within 1 minute of arc \(0.00029\) radians.

12. A refracting surface of radius \(r\) enclosing a medium of relative refractive index \(n\) has its aperture restricted by a narrow circular diaphragm in contact with it. Give an expression for the distance, measured from the surface, of the sagittal images of an object field of infinitely distant stars in terms of the (small) angular distance from the axis. Hence give an expression for the radius of the sagittal image field.

13. Consider a thin lens of negative power — 10 D, and construct a diagram corresponding to Fig. 17 showing the relation between the conjugate distances for a wide range of positions of the object, both real and virtual. Add also the straight lines showing the magnifications for the same range of object distances.
CHAPTER II

The Paraxial Theory of Optical Systems

The discussions in the previous chapter showed the severe limitations which may be associated with the concept of conjugate planes of optical systems. They must not be forgotten in reading the matter to follow.

Collinear Relations

The relations between the object and image spaces valid for the paraxial regions of a system of coaxial refracting surfaces is an imperfect example of a much more general geometrical correspondence theoretically possible between points, lines, and planes in related spaces; this is called "collinear correspondence." Its relevance to an actual optical system will be closer if the system is designed to give sharp images over plane surfaces, without such severe restrictions as hitherto described on the apertures and sizes of the fields. Collinearity is, however, a purely geometrical conception in essence. A geometrical account of the theory was given by J. Clerk Maxwell (Collected Papers, Vol. I, p. 271). The discussion is simplified if it is assumed that the related spaces have a common axis of symmetry, so that if any ray in the one space lies in a plane containing the axis, the corresponding path in the related space will therefore also be in an axial plane. A perfect instrument would then have (as Maxwell showed) the following properties—

(i) Every ray of a pencil originating from a single point of the object space will pass through a single conjugate point of the image space.

(ii) If all the object points lie in a plane surface perpendicular to the axis of symmetry, the conjugate points will also lie in a plane surface perpendicular to the axis in the image space.

(iii) The configurations of the image points of such plane surfaces as mentioned in (ii) will be precisely similar to those of the object points.

Maxwell’s method of discussion is to show that, if perfect correspondence is possible for two such pairs of conjugate planes perpendicular
to the axis, it will be possible for all other such pairs of planes; hence the whole concept is at least a geometrical possibility. Nothing is proved about optical systems as such; the instrument might be represented by some mechanical contrivance. But generations of optical students have found the theory a convenient framework on which to hang their early ideas of the ways in which optical instruments should behave, even if it has to be discarded when it has served its purpose.

The geometrical arguments of Maxwell’s paper, though straightforward, are somewhat lengthy, and it is convenient to use an analytical argument on lines given by Salmon.* He showed that a unique point-to-point correspondence between two spaces in which conjugate points are given by their Cartesian coordinates \(x, y, z,\) and \(x', y', z',\) respectively, is ensured by the relations

\[
x' = \frac{a_1 x + b_1 y + c_1 z + d_1}{ax + by + cz + d} \quad \ldots \quad (2.01)
\]

\[
y' = \frac{a_2 x + b_2 y + c_2 z + d_2}{ax + by + cz + d} \quad \ldots \quad (2.02)
\]

\[
z' = \frac{a_3 x + b_3 y + c_3 z + d_3}{ax + by + cz + d} \quad \ldots \quad (2.03)
\]

in which image space coordinates are distinguished by the accent. If these equations are rearranged and written as a set of linear equations in \(x, y,\) and \(z,\) they may be solved (using the usual determinant theory), and the result appears in the form

\[
x = \frac{a'_1 x' + b'_1 y' + c'_1 z' + d'_1}{a'x' + b'y' + c'z' + d'} \quad \ldots \quad \text{etc.} \quad (2.04)
\]

\[
x = \frac{a'_2 x' + b'_2 y' + c'_2 z' + d'_2}{a'x' + b'y' + c'z' + d'} \quad \ldots \quad (2.05)
\]

\[
x = \frac{a'_3 x' + b'_3 y' + c'_3 z' + d'_3}{a'x' + b'y' + c'z' + d'} \quad \ldots \quad (2.06)
\]

with corresponding equations for \(y\) and \(z,\) where the accented coefficients are functions of the unaccented coefficients in the first set of equations. There is thus a unique point-to-point correspondence.

Further, a plane in the object space is given by the usual analytical expression

\[
lx + ny + nz = p
\]

Substituting from the second set of equations, we obtain a new linear expression in \(x', y',\) and \(z',\) which represents a unique plane in the image space.

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It should be noted that if the object plane has the equation (see (2.01))
\[ ax + by + cz + d = 0 \]
a point conjugate to any point in this plane will, in general, have infinite coordinates, so that the conjugate plane is at infinity. Correspondingly, the image plane, see (2.04),
\[ a'x' + b'y' + c'z' + d' = 0 \]
is seen to be conjugate to an infinitely distant object plane.

These two special planes represent the "principal focal planes" of the system.

The existence of uniquely corresponding planes implies the existence of uniquely conjugate straight lines, since the intersection of two planes represents a straight line. No question of axial symmetry has, so far, arisen.

Such considerations are sufficient to demonstrate the possibility of collinear correspondence, but the property of distortionless imagery requires further argument and we shall now, for the sake of brevity, consider only the case of axial symmetry.

If the axis of symmetry is the z-axis (Fig. 25), any plane perpendicular to the axis will be given by—
\[ z = \text{constant} \]
Hence in our case of axial symmetry the principal focal planes will be given by \( cz + d = 0 \) and \( c'z' + d' = 0 \), respectively. Moreover, if we consider any pair of conjugate planes perpendicular to the axis, the value of \( z' \) cannot be a function of \( x \) or \( y \), hence \( a_3 = b_3 = 0 \), and the equation (2.03) now reduces to the form
\[ z' = \frac{cz + d}{cz + d} \]
If YZ and XZ reference planes for the Cartesian coordinates (as well as the axis) are considered to extend through the instrument, and an object point in the plane \( x = 0 \) has its conjugate point in the same plane, then \( x' = 0 \); also similarly if \( y = 0 \), \( y' = 0 \). These conditions can only be secured in general if the equations (2.01) and (2.02) take the respective forms
\[ x' = \frac{a_1x}{cz + d} \]
\[ y' = \frac{b_2y}{cz + d} \]
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It is evident, however, that a difference between the values of the coefficients $a_1$ and $b_2$ above would be incompatible with axial symmetry and they must in our case be identical. Then for a fixed value of $z$, i.e. for an object plane perpendicular to the axis, there

![Diagram of a space with common axis of symmetry and parallel reference planes.](image)

Fig. 25. Spaces with common (z) axis of symmetry, and parallel reference planes.

must be a constant ratio between the $x$- and $y$-coordinates of conjugate points. In other words, the image formation is free from distortion, and the image is exactly similar to the object.

It is implicit in the foregoing theory that a change in the distance of the object plane from the instrument will in general be accompanied by a corresponding change in the sharp image plane, and also

![Diagram of a ray path.](image)

Fig. 26.

by a change in the magnification. The formal proof can be left to the reader. However, since the foregoing results have been established for three-dimensional space, the discussion of the magnification relations can be conveniently done with the aid of diagrams which show the courses of "rays" situated entirely in axial planes.

Fig. 26 suggests an "instrument" with its axis of symmetry. A ray can, of course, travel along the axis; another incident ray $R$ is shown parallel to the axis, together with its path $R'$ after transmission; the virtual parts of these paths are shown by broken lines.

In general, the instrument causes a change in the angle between
a ray such as R and the axis; if not, it is called a "telescopic" system, a case which will be discussed below. Suppose that an object of constant height in a plane perpendicular to the axis continually alters its position, the locus of the tip of the object being the ray $R$, while the locus of the conjugate image tip is the refracted path $R'$; but the images of axial points remain on the axis. The object (virtual if necessary) could in theory be placed at any distance from the instrument, and owing to the interchangeability of conjugate points the image may thus vary in its distance from "plus infinity" to "minus infinity"; it follows that any real value, positive or negative, of the magnification is possible.

The image point can be completely determined by tracing two rays in an axial plane from the object point through the instrument; since they will lie in the same plane after transmission they must cut in one point or be parallel. In the latter case the image lies at infinity, but in any other case the intersection of the rays determines a unique image plane. Though the opposite case is possible in geometrical theory, it will be assumed (in accordance with the optical case of a refracting instrument) that while the object travels in one direction along the axis the image progresses in the same sense.

Supposing now that the following planes can be found for some given instrument (see Fig. 27).

(i) The object plane (axial point $F$) for which the conjugate image plane is at infinity. The point $F$ is known as the principal focus of the object space.

(ii) The image plane (axial point $F'$) for which the conjugate object plane is at infinity. The point $F'$ is known as the principal focus of the image space.

(iii) The pair of conjugate planes with axial points $P$ and $P'$ respectively for which the magnification is unity. The points $P$ and $P'$ are known as the principal points, and the associated planes as the principal planes.

It is then possible to find, by a simple geometrical construction, the image corresponding to any given object point in the diagram.
such as B. For this purpose two rays are traced; one B₁P₁ parallel to the axis, and one from B₁ through F, cutting the first principal plane in the points P₁ and P₀ respectively. The points conjugate to P₁ and P₀ lie in the second principal plane (owing to the unit magnification) at equal distances from the axis; they are P₁′ and P₀′ respectively; and any ray initially directed towards a particular point in the first principal plane must leave the system in a path which (produced if necessary) passes through the conjugate point.

Further, the two rays B₁P₁ and BP may be regarded as derived from one axial point at infinity; the conjugate point is F′. Hence the final path of B₁P₁ is P₁′F′.

Also the object point F has an image on the axis at infinity, and both the rays FP and FP₀ must, after transmission, be directed to such a point; accordingly, the final path of FP₀ is a line through P₀′ parallel to the axis; this crosses the ray P₁′F′ in the point B₁′ which by the preceding theory must be the unique image point of B₁. It is situated in the plane through B′ drawn perpendicular to the axis and is thus conjugate to the object plane B₁B also perpendicular to the axis.

In order to trace any ray CP₁ through the system it is produced to cut the first principal plane in P₁ (Fig. 28). The point P₁′ on the transmitted path is then known as above. Draw then a ray FP₂ through F parallel to the initial ray; then this pair of rays belongs to a point at infinity, of which the conjugate point Q is in the second principal focal plane through F′, and is uniquely determined since the final path of FP₂ is P₂′Q parallel to the axis where P′P₂′ is equal to PP₂. The final path of the ray CP₁ through P₁ is thus P₁′Q.

The foregoing geometrical constructions lend themselves to the deduction of a number of useful formulae, but it is essential to realize the strict limitation under which they apply to paraxial optics. Diagrams such as Fig. 27 can be looked on as applying
generally to the optical case (as mentioned above) if the *vertical scale is regarded as greatly enlarged* so that the actual heights of objects and images and the apertures of surfaces can be regarded as confined within the paraxial limits. However, various actual systems, such as highly-corrected photographic lenses, approximate in their performance to the collinear relationship between object and image spaces; the imperfections are discussed in the theory of aberrations. Thus, while in the strict optical sense there are, in general, no such things as “principal planes,” such concepts are a useful aid in obtaining a first rough idea of the way in which an optical system performs.

The following relations imply the use of the standard sign convention. The distance (PF = f) from P to F is known as the focal length of the object space (more briefly as the “first focal length”). The distance (P'F' = f') is the focal length of the image space, or the “second focal length.”

In Fig. 27 the distance of the object from the first focal point is FB = x; from the first principal point is PB = l. Similarly F'B' = x' and P'B' = l'. The height of the object BB₁ = h, and of the image B'B₁' = h'. It should be noted that in Fig. 27 x, l, and h' are numerically negative.

The derivation of the following relations from the figure is obvious. We first have alternative expressions for the linear transverse magnification m—

\[ m = \frac{h'}{h} = -\frac{f}{x} \]  
\[ m = \frac{h'}{h} = -\frac{x'}{f'} \]

These equations are often called Abbe’s equations. The above two values for the magnification when equated give—

\[ xx' = ff' \]  

This last relation is called Newton’s equation, since the relation was known to him. Since

\[ x = l - f \]

and

\[ x' = l' - f' \]

the substitution of these values of x and x' gives, on reduction,

\[ lf' + lf = ll' \]

or, on dividing by ll'

\[ \frac{f'}{l'} + \frac{f}{l} = 1 \]
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The conjugate-distance relation for a thin lens is at once recovered if it is recognized that the thin lens will approximate to the case of an optical system with coincident principal planes, and that in this case \( f = -f' \).

Equations (2.09) and (2.10) represent specially simple forms of the conjugate-distance relation which arise through the reference of the object and image distances to special origins. For example, take an object-space origin at a distance \( x_0 \) from the first principal focus, and an image-space origin at a distance \( x_0' \) from the second. Then let conjugate distances measured respectively from these new origins be \( b \) and \( b' \) so that—

\[
\begin{align*}
x &= b + x_0 \\
x' &= b' + x_0'
\end{align*}
\]

Substitution of these values in (2.09) gives

\[
bb' + b'x_0 + x_0'b + x_0x_0' = ff' \quad . \quad (2.11)
\]

Suppose that \( m_0 \) is the linear magnification for the case of an object at the new origin for its space and that \( m_0' \) is the linear magnification when an image is at the new origin for the image space, we find on dividing (2.11) by \( ff' \) that

\[
\frac{bb'}{ff'} - \frac{b'}{f} \frac{1}{m_0} - \frac{b}{f} \frac{m_0'}{m_0} + \frac{m_0'}{m_0} = 1 \quad . \quad (2.12)
\]

(2.11) and (2.12) are examples of a more general type of conjugate-distance relation which will apply, for example, when such distances are measured from the surfaces of a lens.

The Focal Lengths and the Relation between Them

As the name implies, a “focal length” of a system is a linear magnitude defined (as above) in the paraxial theory; its practical importance lies in the fact that it is often the factor which determines the size of image given by a system. For example, Fig. 29 suggests a pair of parallel rays which might be two of a complete bundle of rays derived from an object point at an infinite distance, and thus have a conjugate image point in the second principal focal plane through \( F' \); the angle between all the incident rays and the axis is \( \omega \). The size of the image can be found by tracing the ray through \( F \) which cuts the first principal plane in \( P_1 \) and leaves the second principal plane from \( P_1' \) at the same height \( (h') \) from the axis, to which its final path is parallel. The image height is \( h' \), where, in the paraxial case,

\[
h' = f\omega \quad . \quad . \quad . \quad . \quad (2.13)
\]
(Note that by the sign convention \( h' \) is positive in the diagram while both \( f \) and \( \omega \) are numerically negative.)

A useful general definition of the focal length \( f \) is thus implied. Granted that the dimensions of the image are measured in the principal focal plane of the image space, the object-space focal length is the limiting value, when the distance of any part of the image from the axis tends towards zero, of the quotient of the distance between two image points divided by the angular subtense of the interval between the corresponding infinitely distant object points. Note that the definition will hold if the image extends on both sides of the axis, or indeed if the image points are not in the same axial plane, since the scale of magnification is constant for the paraxial region. If the proper sign

![Diagram](image)

**Fig. 29.**

coventions are taken into account an inverted image has a height of the sign opposite to that of the object.

The formal justification of a corresponding definition of the second focal length \( f' \) may be left to the student. The focal length of the image space is the limiting value, when the distance of any object point from the axis tends towards zero, of the quotient of the linear distance between two object points in the principal focal plane of the object space divided by the angular subtense of the interval between their infinitely distant images. The corresponding equation is, of course,

\[
f' = \frac{h'}{\omega'}
\]  

(2.14)

In Fig. 29, \( f \) and \( f' \) are PF and P'F' respectively. A relation between them is given by applying the Smith–Helmholtz relation (1.41) to the case of the object and image at the conjugate points \( P \) and \( P' \), where the object and image heights are equal and thus cancel from the equation.

Accordingly, for these points* and a ray path intersecting them,

\[
n\omega = n'\omega'
\]

* The introduction of \( h' \) in this case is inadvisable because it has been used to indicate \( F'B_i' \).
However, the angle $\omega$ in the paraxial case is $PP_1/ff'$, and the angle $\omega'$ is $F'B_1'/ff'$. Hence, since $PP_1$ and $F'B_1'$ are equal,

$$\frac{n}{f} = -\frac{n'}{f'} \quad \cdot \quad \cdot \quad \cdot \quad (2.15)$$

Thus the focal lengths, while of opposite sign, have the same numerical ratio as the refractive indices of their respective spaces; in the special case of a system in air they are numerically equal.

**General Conjugate Distance Relations**

The conjugate distance relation (2.10) for a system referred to its principal planes as origins can now be written in a simplified form with the help of (2.15), i.e.

$$\frac{n'}{l'} - \frac{n}{l} = \frac{n'}{f'} = -\frac{n}{f} \quad \cdot \quad \cdot \quad \cdot \quad (2.16)$$

*If the system is in air* the reduced form is

$$\frac{1}{l'} - \frac{1}{l} = \frac{1}{f'} = -\frac{1}{f} \quad \cdot \quad \cdot \quad \cdot \quad (2.17)$$

**Calculation of the Focal Length from the Data of a System**

If the data of a system of coaxial refracting surfaces are given in terms of the vertex radii, the refractive indices, and the separations, the focal lengths may be calculated if the successive conjugate distances are calculated surface by surface for an initial object at infinity in order to get the final image height. Thus for the first surface, we have from (1.19)

$$l_1' = \frac{n_1' r_1}{n_1' - n_1}$$

If the separation between the first and second surfaces is $d_1'$ we have

$$l_2 = l_1' - d_1'$$

also

$$\frac{n_2'}{l_2'} - \frac{n_1'}{l_2} = \frac{n_2' - n_1'}{r_2}$$

and so on for all the surfaces. If the angular subtense of the object is $\omega$, the size of the image $h_1'$ formed by the first surface is given by $f_1\omega$ (from equation 2.13) where $f_1$ is $-n_1' r_1/(n_1' - n_1)$, i.e.

$$h_1' = -\frac{n_1' r_1 \omega}{n_1' - n_1}$$
Further, we have from (1.39)

\[
\frac{h'_2}{h'_1} = \frac{l'_2/n'_2}{l_2/n_2}
\]

so that if the system contains \( k \) surfaces, and the symbol \( \Pi \) denotes the product of all the terms,

\[
\frac{h'_k}{h'_1} = \Pi_{j=2}^{j=k} \frac{l'_j/n'_j}{l_j/n_j}
\]

Thus

\[
h'_k = -\frac{n_2 r_1}{n_1' - n_1} \frac{k}{2} \frac{l'_2/n'_2}{l_2/n_2}
\]

However, the final image height is \( f'\omega \), where \( f \) is the first focal length of the entire system. Hence

\[
f = -\frac{n_2 r_1}{n_1' - n_1} \frac{k}{2} \frac{l'_2/n'_2}{l_2/n_2}
\]

The value of \( f' \) is, of course, easily calculated from (2.15), or the above procedure could be applied in the reverse order.

**Power of a System**

In the case of a simple system of two or three surfaces or lenses, useful explicit formulae may be derived (as in the work of Gauss) for the power of the whole system in terms of the powers of the components. The formula expressing the change in the inclination of a paraxial ray incident at a height \( y \) in a surface of power \( F \) is, from (1.18),

\[
n'u' - nu = yF
\]

Referring to Fig. 30, let \( P_1 \) and \( P_2 \) be the points of intersection of a paraxial ray with the refracting surfaces whose poles are \( A_1 \) and \( A_2 \) on the axis of symmetry. The perpendiculars dropped from \( P_1 \) and \( P_2 \) to the axis are shown. However, if \( r \) be the radius of curvature of a surface and \( y_0^2/2r \) is negligible\(^*\) at both \( A_1 \) and \( A_2 \), the interval between the feet of the perpendiculars will be indistinguishable from \( d_1' \), the axial separation \( A_1A_2 \).

The angle between the ray and the axis after the first refraction is \( u'_1 \). The refractive indices of the media from left to right are written as \( n_1, n_a, \) and \( n_2' \) respectively. Refraction at the first surface is now expressed by

\[
n_a u'_1 - n_1 u_1 = y_1 F_1
\]

\(^*\) See the argument on p. 30.
In the paraxial limit,

\[ y_2 = y_1 - d_1'u_1' \]

Refraction at the second surface is expressed by

\[ n_2'u_2' - n_2u_1' = y_2F_2 \]

Eliminating \( y_2 \) and \( u_1' \) between these last equations, a relation can be obtained which expresses the final angle of convergence of the ray, \( u_2' \), in terms of the data of the initial ray.

Before writing down the equation, however, it is worth-while to explain and use a notation used by Gauss, who was one of the first to use equations of this type. It will be noted in the preceding sections that the angles often occur with the corresponding refractive index as a multiplying factor, and that axial lengths often appear as divided by the refractive index. In order to obtain compact expressions, the use of bold-face type for a symbol of an angle or a length will indicate the presence of the refractive index in the senses just mentioned; thus, for example,

\[ I = l/n, \quad u = nu \]

Such lengths or angles are sometimes known as "reduced" quantities. The three expressions last given can now be written in the form

\[ u_1' - u_1 = y_1F_1 \quad \cdot \quad \cdot \quad \cdot \quad (2.19) \]
\[ y_2 = y_1 - d_1'u_1' \quad \cdot \quad \cdot \quad \cdot \quad (2.20) \]
\[ u_2' - u_1' = y_2F_2 \quad \cdot \quad \cdot \quad \cdot \quad (2.21) \]

Note that the \( y \)-values (or any lengths measured perpendicular to the axis) appear unchanged in the equations.

Eliminating \( y_2 \) and \( u_1' \) the following expression is obtained—

\[ u_2' = y_1(F_1 + F_2 - F_1F_2d_1') + u_1(1 - F_2d_1') \quad (2.22) \]

In order to find the image space focal length of any system, a simple procedure is to trace through it (by any convenient means)
a paraxial ray incident parallel to the axis at a given height \( y_1 \); see Fig. 31, in which the optical system is merely symbolic of some possible case. If the final angle between the ray and the axis is \( u' \) then the focal length \( f' \) of the image space is \( y_1/u' \). Thus the reduced focal length is

\[
f' = \frac{y_1}{u'}
\]

Returning to our special problem, if the condition that \( u_1 = 0 \) is inserted in (2.22),

\[
\frac{y_1}{u_2'} = f' = \frac{1}{\frac{F_1}{F_2} + F_2 - F_1 F_2 d'}
\]  
(2.23)

(where \( d \) is written instead of \( d' \), for this two-surface problem).

![Fig. 31.](image)

Note that if the incident ray is parallel to the axis; then, for the complete system,

\[ u_2' = y_1 (F_1 + F_2 - F_1 F_2 d) \]

Comparing this with the corresponding equation for a single surface,

\[ u_1' = y_1 F_1 \]

it is evident that the expression in brackets is analogous to the power of the surface; it is accordingly called the power, \( F \), of the system. Thus

\[ F = F_1 + F_2 - F_1 F_2 d = n_2'/f' \]  
(2.24)

The equation (2.22) can now be written—

\[ u_2' = y_1 F + u_1 (1 - F_2 d) \]  
(2.25)

and is readily used to ascertain the position of the first principal focus with respect to the first surface of the pair. If we trace (Fig. 32) a paraxial ray through the first principal focus at an angle
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\( u_1 \) with the axis it will emerge parallel with the axis, so that \( u_2' \) is zero. Then since the distance

\[
A_1F = \frac{y_1}{u_1} = \frac{n_1y_1}{u_1}
\]

we obtain from (2.25) just above, putting \( u_2' = 0 \),

\[
\frac{A_1F}{n_1} = -\frac{1 - F_2d}{F} \quad \ldots \quad (2.26)
\]

The relation (2.15) between the focal lengths gives with (2.24) above

\[
\frac{f}{n_1} = \frac{PF}{n_1} = -\frac{1}{F} \quad \ldots \quad (2.27)
\]

so that, subtracting the corresponding sides of the last two equations

\[
\frac{PF - A_1F}{n_1} = \frac{PA_1}{n_1} = -\frac{F_2d}{F}
\]

or

\[
PA_1 = -\frac{n_1F_2d}{F} \quad \ldots \quad (2.28)
\]

If the system is imagined to be reversed, the corresponding equation is easily deduced from consideration of symmetry or otherwise—

\[
P'A_2 = \frac{n_2'F_1d}{F} \quad \ldots \quad (2.29)
\]

so that the distances of the apices of the surfaces from the respective principal points of the system are easily calculated.

**Thick Lens in Air**

If the system consists of a thick lens of refractive index \( N \) in air of refractive index assumed to be unity, the power \( F \) of the complete lens is expressed in terms of the powers \( F_1 \) and \( F_2 \) of the surfaces and the reduced thickness \( d/N \) as follows—

\[
F = F_1 + F_2 - F_1F_2(d/N)
\]
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The distances of the principal points from the apices of the surfaces are calculated from the expressions—

\[
\begin{align*}
A_1P &= \frac{F_2(d/N)}{F} \quad \ldots \quad (2.30) \\
A_2P' &= -\frac{F_1(d/N)}{F} \quad \ldots \quad (2.31)
\end{align*}
\]

These equations are easily used to compute the position of the principal points in lenses of differing shapes. If the total power remains constant while the powers of the surfaces are varied relatively to each other the effect is equivalent to the imaginary "bending" of a lens, as suggested in Fig. 33. The distance of a principal point from a surface is seen, from (2.30), to be directly proportional to the thickness of a lens (other things being equal), so that lenses of negative power, in which the thickness is a minimum on the axis, do not exhibit the relatively marked variations of principal point position which occur on bending, with the usually thicker positive lenses. Fig. 33 should also lend itself to the realization of the very attenuated space which can properly be called paraxial. In a plano-convex lens one "principal plane" coincides with the curved surface; it is only within the region that these two surfaces have a negligible separation that the paraxial relations are adequate.

System of Three Surfaces

The relations (2.20) and (2.22) given above for a system of two surfaces may be written in the form

\[
\begin{align*}
y_2 &= y_1(1 - d_1'F_1) - u_1d_1' \\
u_2' &= y_1F_{12} + u_1(1 - F_2d_1') \quad \ldots \quad (2.32)
\end{align*}
\]

where \(F_{12} = F_1 + F_2 - d_1'F_1F_2\). If a third surface of power \(F_3\) is added at a reduced distance \(d_2'\) from the second surface, the
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following equations can be added—

\[
\begin{align*}
y_3 &= y_2 - d'_2 u'_2 \\
u'_3 &= y_3 F_3 + u'_2
\end{align*}
\]

(2.33)

Eliminating \(y_3\) and \(u'_2\) between (2.32) and (2.33), the following relations, cast into a form similar to (2.20) and (2.25) above are obtained by straightforward algebra.

\[
\begin{align*}
y_3 &= y_1(1 - d'_1 F_1 - d'_2 F_{12}) - u_1[d'_1 + d'_2(1 - F_2 d'_1)] \\
u'_3 &= y_1[(1 - d'_1 F_1 - d'_2 F_{12}) F_3 + F_{12}] + u_1[(1 - F_2 d'_1) - F_3[d'_1 + d'_2(1 - F_2 d'_1)]]
\end{align*}
\]

(2.34)

These equations are also analogous to (2.32) and it is clear that the power of the system \(F_{123}\) is the coefficient of \(y_1\). Written in entirety it is

\[
F_{123} = F_1 + F_2 + F_3 - d'_1 F_1 F_2 - (d'_1 + d'_2) F_1 F_3 - d'_2 F_2 F_3 + d'_1 d'_2 F_1 F_2 F_3 .
\]

(2.35)

Also, as before, the reduced distance \(- A_1 F / n_1\) corresponding to the interval between the apex of the first surface and the principal focus of the object space is found by dividing the coefficient of \(u_1\) in the second of equations (2.34) by the coefficient of \(y_1\); thus

\[
\frac{A_1 F}{n_1} = - \frac{1 - F_4 d'_1 - F_3 d'_1 - F_3 d'_2 + F_3 F_3 d'_1 d'_2}{F_{123}} .
\]

(2.36)

and so on; the further results may be derived as a useful exercise.

**System of Two Thin Lenses**

The analysis in the section dealing with two separate surfaces lends itself immediately to dealing with the case of two thin coaxial lenses in air. If the “deviation” equations are written down, first for a surface and next for a complete thin lens, as follows—

Single surface:

\[ u' - u = y F \]

Single lens:

\[ u' - u = y F \]

it will be seen that the procedure in dealing with the lenses instead of the surfaces will be the same, except that the “reduced” distances will be replaced by ordinary distances; thus \(d\) will be replaced by \(d\), the separation of the two thin lenses. Hence in this case

\[
F = F_1 + F_2 - F_1 F_2 d .
\]

(2.37)

and the expressions for the position of the principal points relative
to the constituent lenses are obtained by putting \( n = 1 \) in equations (2.30) and (2.31).

**Numerical Example.** Consider the case of a positive thin lens of power 10 D, followed at a distance of 5 cm by a negative thin lens of power \(- 10\) D.

The power of the system is

\[
F = 10 - 10 - (10)(-10)(0.05) = 5 \text{ D}
\]

where the distances are expressed in metres in accordance with the use of the dioptric scale of power. Thus in spite of the equal positive and negative powers of these lenses, which would tend to annul each other if the separation could be reduced to zero, the finite separation has produced a system of substantial power with a focal length of 20 cm. It is an example of a "telephoto" lens.

The distance of the first principal point from the first lens is easily calculated to be \(-10\) cm, and this is also equal to the distance of the second principal point from the second lens. The principal planes are thus relatively situated as suggested in Fig. 34, and the distance \( P'F' (= 20\) cm), which is the second focal length, is double the actual separation of the last lens from the principal focal plane. Thus the size of the image is much greater than could otherwise be obtained with a camera of the same length. The "telephoto effect" is a phrase sometimes used for the ratio of the focal length of the combination to the "back focal length," i.e. the distance between the rear lens and the principal focus; in this numerical case the ratio is 2.

**Relations between Object and Image Spaces; Axial Magnification**

Equations (2.07), (2.08), and (2.09), when applied to a truly collinear relationship between spaces enable the Cartesian equation of the section of a surface by an axial plane, referred to the principal focus of one space as origin, to be directly transformed into the equation of the conjugate curve in the image space and referred to the other principal focus. Thus for example if the section is a straight
line, where $M$ is the tangent of the angle of slope and $C$ is a constant, its equation may be written—

$$h = Mx + C$$

Substitution of the image space variables yields

$$h' = -x'(C/f') - Mf$$

which is, of course, another straight line. It will be evident that the degree of an equation will remain unchanged, but that in general the character of a curve will be altered, so that a circle will be liable to be transformed into an ellipse or hyperbola. With regard to three-dimensional objects it may be inferred that if there is an object sphere symmetrical about the axis it may have a conjugate ellipsoid or hyperboloid of revolution; thus the relationship between the spaces exhibits distortion of this character, though planes remain conjugate to planes, as is implicit in the Maxwellian theory.

The strictly local axial magnification relating the axial shift of an image point to a corresponding axial shift of the object may be found by differentiating Newton's equation (2.09). Thus—

$$\frac{dx'}{dx} = -\frac{x'}{x}$$

and by multiplying together the Abbe equations (2.07) and (2.08), and writing $m$ for the magnification $(h'/h)$, we obtain

$$\frac{x'}{x} = m^2 \frac{f'}{f}$$

$$\frac{dx'}{dx} = m^2 \left( \frac{n'}{n} \right)$$

(2.38)

**Perspective**

If a "picture" of solid objects is to be readily interpretable, the relative angles of view presented by the parts of the picture to the viewing eye should be similar to those which would be obtained if the eye should view the corresponding objects themselves. A plane picture is thus usually a "perspective projection," with respect to some centre. The eye then obtains true perspective from the picture only if it is placed in the same relative position as this centre in regard to the picture. The idea is illustrated in Fig. 35, where $P$ is the perspective centre. Apparent space distortion through false perspective is discussed in the present writer's book on Technical Optics.*

The Nodal Points

In regard to the two preceding paragraphs, it is of significance to inquire whether points in the object space and image space respectively can be found such that the perspective relations of the elements in the object space with regard to the first point will be identical with those of the elements of the image-space with regard to the second. The existence of conjugate points on the axis, having such properties, the nodal points, was shown by Listing. The discussion below assumes perfect collinear correspondence.

The characteristic property of such points must be that an entrant ray directed towards the first nodal point at a certain angle with the axis must leave the system as if it came from the second nodal point at the same angle with the axis.

Fig. 36 shows a system with principal focal planes at F and F'. Take any point K in the second focal plane, and draw a ray KP' cutting the second principal plane in P'. Its path after transmission is P'F, where PP' = P'P' = F'K. Now draw KNP' parallel to P'F, cutting the axis in N' and the second principal plane in P'; traced through the system it must pass through P where PP' = P'P'; and moreover it must emerge parallel to P'F since it is derived from K in the second principal focal plane (and K is
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conjugate to an object point at infinity); the path of this refracted ray cuts the axis in N.

It is implicit in this construction that the angle $\hat{\text{FPN}_0}$ is equal to the angle $\hat{\text{KN}F'}$; thus if N and N' should have positions independent of the height of K, they would have the properties of nodal points. This must be so, since the triangles $\text{FPP}_1$ and $\text{N'F'K}$ are equal in all respects by the above construction; thus

$$PF = F'N' = f$$

Moreover, the ray $\text{NP}_0$ cuts the first focal plane in $\text{F}_0$, so that since $\text{FF}_0 = \text{F}_1'\text{P}_0 = \text{P}_1'\text{P}_0'$, the triangles $\text{F}_0\text{FN}$ and $\text{P}_0'\text{P}_1'\text{K}$ are equal in all respects by the construction. Therefore $\text{FN} = \text{P}_1'\text{K}$; but $\text{P}_1'\text{K} = \text{P}F'$ and thus

$$\text{FN} = \text{P}F' = f'$$

It is also clear from the parallelogram $\text{P}_0'\text{P}_0\text{N'}\text{N}$, that the separation $\text{NN}'$ of the nodal points is equal to that of the principal points PP'.

The above figure is drawn for a case where the focal lengths are numerically unequal. If the refractive indices of the initial and final spaces are equal, the focal lengths become equal to each other and the above equations show that the nodal points N and N' coincide respectively with P and P'.

The limits under which the above theory is valid for an optical system will follow from the discussion of the general applicability of collinearity (see p. 49).

The general Smith–Helmholtz relation valid for the paraxial optical case shows, if applied to the conjugate nodal planes through N and N' where the angles $u$ and $u'$ are equal, that the magnification $(h_N'/h_N)$ for the nodal planes is $n/n'$. Moreover, the extended relation $nh\beta = n'h'\beta'$ implies that if any pair of entrant paraxial rays are directed towards a point in the first nodal plane, they will leave the system as diverging from the conjugate point in the second nodal plane, with the same angle between them.

Conjugate Distance Relations Referred to the Nodal Planes

If $z$ and $z'$ denote respectively the distance of object and image planes from the corresponding nodal planes then, employing the other symbols as above,

$$z = x - \text{FN}$$
$$= x - f'$$

since

$$z' = x' + \text{N'F'}$$
$$= x' - f$$
Hence, substituting in Newton’s equation (2.09) and simplifying,
\[ zz' + z'f' + fz = 0 \]
or
\[ \frac{f}{z'} + \frac{f'}{z} + 1 = 0 \]
Using (2.15), the above is readily transformed into
\[ \frac{1}{n'z'} - \frac{1}{nz} = \frac{F}{nn'} \]
a type of relation on which it is possible to build an account* of optical theory—parallel to the Gaussian Theory—but in which we encounter products of lengths and refractive indices.

Notes on Focal-length Measurements

The direct determination of the distance between the surface of a lens and its principal focus is sometimes adequate for finding roughly the focal length of thin positive spectacle lenses. In the general case, however, this “back focal distance” is very different from an actual focal length as defined on page 52 above. The definition suggests possible experimental means useful for a photographic lens.

1. The angular subtense of a pair of stars or other very distant pair of well-defined objects is found with a theodolite or similar instrument. The distance between their real images (which should both be close to the axis of the lens) is then found by some convenient means, for example by taking a photograph with the lens and measuring the plate. The distance between the image points is divided by the angular subtense in radians. Strictly, the paraxial focal length† is the limit of this quotient when dividend and divisor become smaller and smaller.

2. The actual distant objects may be replaced by “cross-lines” or other marks placed in the principal focal plane of an auxiliary lens and thus constituting a “focal collimator.” The images projected by this lens are “at infinity” and the essential idea of the procedure is as before. If the system under test has a very short focal length, for example an eye-piece, the images may often be measured conveniently with a measuring microscope.

3. If the preceding methods are impracticable it is sufficient to observe the magnification (with the aid of a suitable optical bench

† The “focal length” of a camera lens may convey a somewhat different meaning where wide-angle fields are concerned. See Hotine, M., Pro. Pap. Air Sur. War Off., No. 5.
and micrometers) for two pairs of conjugate positions, observing at the same time the distance between the two positions either of the object or the image plane.

Writing the Abbe equation (2.08) in the form

\[ \frac{x_1'}{f} = -m_1 \]

where the suffix 1 refers to the first position of observation, the magnification \( m_2 \) in the second position is represented by

\[ \frac{x_2'}{f} = -m_2 \]

Subtracting these two equations, the value of \( f' \) can be obtained as

\[ f' = \frac{x_2' - x_1'}{m_1 - m_2} \]

that is, the quotient of the distance between the two positions of the image divided by the difference between the two magnification ratios. The corresponding equation for use when the change in the object distance is observed should be written down by the student.

4. A method which obviates the actual measurement of images or objects (although at the expense of some necessary mechanical arrangements) is the so-called "nodal slide" method. A "collimator" (Fig. 37), in which a cross-line is placed at the principal focus of an auxiliary lens, simulates an infinitely distant object. Imagine that the parallel rays therefrom are first incident along the axis of the lens under test, and further that this lens is rotated to and fro through a small angle about its second nodal point. Owing to the

---

**Fig. 37. Theory of the nodal slide method.**
properties of the nodal points, the ray emergent as from the second nodal point cannot now alter its direction or position, so that there can be no lateral movement of the image; but if such rotation takes place about any other axial point, lateral movements of the image will be observed. The method therefore involves a mechanical arrangement whereby the lens can be so rotated about any trial point on its axis in the neighbourhood of the lens; when lateral movement fails to accompany such rotations the axis of rotation passes through the second nodal point. The actual task in measurement is to find the separation between this axis of rotation and the principal focus of the lens. Text-books of practical optics should be consulted.

There are of course numerous other methods of measuring focal lengths; practical details have been omitted from the above notes. It may, however, be noted by the student that similar principles can be applied for the measurement of the focal lengths of negative systems, though in such cases the image formed may be virtual and inaccessible. Thus the complication often consists in the necessity of measuring something inaccessible. Suitable means of doing this in the laboratory, e.g. long-focus measuring microscopes, cathetometers, the projection of a real image under a pre-determined magnification, etc., have therefore to be employed.

**Reflecting Systems**

Optical calculations in respect of systems containing reflecting components can be simplified if reflection is treated as a special case of refraction; but it is essential always to start with the general forms of the refraction equations involving the refractive indices of the spaces. If this be done, any equation symbolic of refraction can be transformed into the corresponding form for reflection by putting \( n' = -n \). Take for example the law of refraction of a ray at a surface, i.e.

\[
n \sin I = n' \sin I'
\]
THE PARAXIAL THEORY OF OPTICAL SYSTEMS

The substitution just mentioned gives

\[ n \sin I = - n \sin I' \]

or

\[ I = - I' \]

which expresses the direction of the reflected ray (compare Fig. 38) in accord with the sign convention, and is a more useful form for the law than a mere statement of the numerical equality of \( I \) and \( I' \).

Consider further the general paraxial conjugate distance equation for refraction at a surface of radius \( r \)

\[ \frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{r} = F \]

The same substitution gives

\[ - \frac{n}{l'} - \frac{n}{l} = - \frac{2n}{r} = F \]

from which it is important to notice that the power of the reflecting surface \((-2n/r)\), which would have to be used in any calculation of the power of a system including it, contains the refractive index of the medium in contact with the surface, and this factor must not be omitted unless the reflecting surface is in air. Of course a simpler form of the last equation is

\[ \frac{1}{l'} + \frac{1}{l} = \frac{2}{r} \]

which is the usual conjugate distance relation for a spherical mirror derived by ad hoc geometry in elementary text-books. Note that if the light is incident from left to right a reflecting surface of positive radius will have a negative power, and vice versa; this is in accord with our system which assigns a positive power to surfaces which produce real images of infinitely distant objects.

Similarly, the astigmatic conjugate relations (1.35) and (1.36) will reduce to

\[ \frac{1}{s'} + \frac{1}{s} = \frac{2 \cos I}{r} \]  \hspace{1cm} \text{(sagittal case)}

and

\[ \frac{1}{l'} + \frac{1}{t} = \frac{2}{r \cos I} \]  \hspace{1cm} \text{(tangential case)}

It is a useful exercise for the student to write ad hoc derivations of the above formula. It is of interest to note that if \( s \) and \( t \) are infinite the equations indicate that the astigmatic images given by a concave mirror, tangential and sagittal, will lie respectively on a sphere of
radius $r/4$ and on a plane surface (see Fig. 39) where these are cut by the reflected ray. These surfaces then are the astigmatic image fields for this case. This result is of immediate practical importance in the theory of the reflecting telescope employing a single mirror as its objective.

![Fig. 39. The astigmatic image fields of a concave mirror.](image)

$$s = t = \infty$$
$$s' = r(2 \cos I)$$
$$t' = (r/2) \cos I$$

**Catadioptric Systems**

Calculations of the powers or focal lengths, etc., in a system involving reflections can be carried out by formulae of the Gaussian type (2.24), if it is remembered that a symbol such as $d_1'$ means the distance from surface 1 to surface 2 in the direction of progress of the light. Thus if this initial direction of progress of the light is from

![Fig. 40.](image)

left to right, $d_1'$ is positive. Now if 2 is a reflecting surface (compare Fig. 40), the direction of the light is reversed so that $d_2'$ will be negative according to the convention. However, the reflection involves the change of sign of the refractive index of the medium, so that the reduced distance $d_2'$ will still remain positive. Let us take the case of a lens silvered on the back; the radii are $r_1$ and $r_2$, and the refractive indices outside and inside respectively are $n$ and $n_1$; the thickness
is $d$. Light is incident from left to right and is refracted before and after reflection. The powers of the first two surfaces are

$$F_1 = \frac{n_1 - n}{r_1}, \quad F_2 = -\frac{2n_1}{r_2}$$

When we consider the final refraction, the general formula $(n' - n)/r$ indicates that we subtract the "index before refraction" from the "index after refraction," i.e. (remembering the reversal of the light) we shall obtain the same "power" as before. The convention for the radius of curvature is not affected. Computed in this way any numerical power of a surface and any reduced thickness is independent of the direction of the light. Thus in our case of the back-silvered lens, $F_3 = F_1$. The power of the whole system can therefore be written down, equation (2.35), as follows—

$$F = 2F_1 + F_2 - 2d F_1 F_2 - 2d F_1^2 + d^2 F_1^2 F_2$$

and the reduced distance from the first surface to the principal focus will be

$$\frac{A_1 F}{n} = \frac{1 - F_2 d - 2F_1 d + F_1 F_2 d^2}{F}$$

Note that if the thickness is negligible, the power of the whole system is $2F_1 + F_2$. If this is written in terms of the radii it is

$$F = 2 \frac{n_1 - n}{r_1} - 2 \frac{n_1}{r_2}$$

$$= 2(n_1 - n) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) - \frac{2n}{r_2}$$

The significance of this last form is worthy of notice.

**EXERCISES II**

1. An object of height 1 metre is situated at a distance of 4 metres from the nearer principal point of an eyepiece which projects an inverted image of height 3.5 millimetres. What is the focal length of the eyepiece?

2. Two stars subtend an angle of 4 minutes of arc. They are photographed by an astrophotographic objective, and their images are separated on the plate by 1 mm. What is the focal length of the objective?

3. The optical system of the eye is sometimes represented simply by a spherical refracting surface of radius 5.55 mm, enclosing a medium of refractive index 4/3. Find the two focal lengths. Find also the retinal separation of the images of two stars subtending an angle of 1° in the visual field.
4. A sphere of radius 10 cm, and refractive index 1.5 is half immersed in a liquid of refractive index 1.75. Find the position of the Gaussian focal planes, unit planes, and nodal planes of the system for narrow central pencils of light.

5. A system of coaxial spherical surfaces separates media of which the first and last have refractive indices \( n \) and \( n' \) respectively. Assuming \( x \) and \( y \) to be the distances of object and image from corresponding foci, and that the product \( xy \) is constant, determine the position of a line, perpendicular to the axis, about which the system may be rotated through a small angle without appreciable movement of the real image of a very distant axial object.

6. Show that for a fairly thin equi-convex lens, the distance between the principal points is about one-third the thickness of the lens if the refractive index is about 1.5.

7. A system consists of two coaxial lenses of negligible thickness. Explain how the focal lengths of both the lenses can be found by measuring the focal lengths of the system for various distances between the components.

8. The data of Gullstrand's schematic eye are (see Fig. 89)—

\[
\begin{align*}
Aqueous humour & \quad r_1 = 0.0078 \text{ m} \\
& \quad n = 1.336 \quad d_1 = 0.0036 \text{ m} \\
\text{Crystalline lens} & \quad r_2 = 0.01 \text{ m} \\
& \quad n = 1.4130 \quad d_2 = 0.0036 \text{ m} \\
\text{Vitreous humour} & \quad r_3 = -0.0060 \text{ m} \\
& \quad n = 1.336
\end{align*}
\]

Calculate the two focal lengths and the positions of the second principal and focal points of the system. The thickness of the cornea may be neglected.

9. An equi-convex lens is made with both refracting surfaces of power 2 D. The thickness is 1.50 cm, and the refractive index of the glass is 1.50. The lens is silvered on the back. Find the power of the catadioptric system and the positions of the Gaussian points.

10. Find the conditions under which the distance between the paraxially conjugate planes for a lens of negligible thickness in air may be a minimum.

11. Starting with Newton's equation, discuss the possibility of determining the focal length of a thick lens with the aid of an optical bench, but without measurements of magnification or image size.

12. Investigate the conditions under which a lens system in air, in which \( d \) is the separation PP' between the principal planes, and \( f' \) is the focal length of the image space, may have an image coinciding in space with the object.

13. A divergent pencil of light is reflected at the convex surface of a spherical mirror of radius of curvature 15 cm, the aperture being very small. The distance of the real object point being 40 cm, and the angle of incidence 60°, find the positions of the sagittal and tangential foci.
CHAPTER III

Optical Instruments

The type of optical instrument to be considered in this chapter is one which is designed to project an image, real or virtual, into some position convenient for observation or recording. It will in general comprise lenses or mirrors of forms symmetrical with regard to an axis,* and of necessity these will usually be supported in "mountings" with circular apertures also symmetrical with respect to the axis. Symmetrical diaphragms with circular apertures, not in contact with any lens, may also be employed in the system.

The Pupils

Consider first the rays from an axial point B of the object; some of these which may succeed in traversing the first lens may be stopped by a diaphragm D encountered afterwards. Fig. 41 suggests such a case. If we imagine that the image EP of the diaphragm D is formed by the antecedent part of the system (in this case the lens L), we may regard EP (which lies theoretically in the object space) as the "entrance pupil" which limits the angular aperture of the cone of rays which can traverse the entire system, if, as in this diagram, there is no further obstacle in the path of any ray from B which is transmitted by D. If there are several such diaphragms or lens mounts, the image of each of these projected by all the antecedent part of the system will in each case subtend some ascertainable angle at the object point; that image which subtends the smallest angle is the effective entrance pupil. If the diameter of the front lens mount (or a diaphragm antecedent to the front lens) is small enough, this aperture itself may constitute the entrance pupil.

Supposing that the entrance pupil has been identified as the image of some particular diaphragm or lens mount in the system; then, to repeat, all the ray paths which can pass this diaphragm will reach the image point. The exit pupil is then the image of this particular diaphragm formed by all the parts of system posterior to it. It will be evident that the entrance pupil and exit pupil occupy conjugate positions with regard to the entire system.

* The direction of the "axis" may suffer deviations through the use of prisms, etc.
If the object point is moved to some other position on the axis, then, since the various diaphragm images mentioned above are not coincident, their relative angles of subtense will vary, and a new entrance pupil (and exit pupil) may come into operation. This may be realized perhaps more clearly with the aid of Fig. 42, in which the supposed diaphragm images are shown disassociated from the system. The same entrance and exit pupils are usually valid for conjugate points near the axis, but if the object point moves very far perpendicular to the axis the limitation of the cone of rays traversing the system is generally subject to the partial occultation of one aperture by another; this effect is known as vignetting, and it may in practice be the principal factor in limiting the size of the image, by restricting more and more the angular apertures of the bundles of rays which can reach image points farther and farther from the axis.

The principal ray from any object point is the one which passes through the centre of the entrance pupil.

**Depth of Focus**

If a focusing screen (Fig. 41) is mounted perpendicular to the axis in the neighbourhood of the axial image point B', the object B being
a bright source of dimensions negligible in comparison with its
distance, the concentration of light on the screen will be greatest
when it lies accurately in the image plane of $B'$, provided that (as
will be supposed here) the optical system is sufficiently free from
aberration.

If the screen is moved, in a direction parallel to the axis, through
a distance $\pm \delta l'$, the diameter $p$ of the ray-patch on the screen will be

$$p = 2\delta l' \tan U'$$

where $U'$ is the semi-angle of the cone of rays passing through the
image.

When a camera is focused with the aid of a ground-glass screen,
unaided vision is not very critical, and the diameter $p$ of the patch
may approach $0.005$ in. or even $0.01$ in., before an appreciable loss
of definition is noticed. The tolerable range is given by

$$\delta l' = \frac{1}{2} p \cot U'$$

and is increased by the reduction of the angle of the cone, as for
example by the use of a smaller "stop" or diaphragm.

If, however, the image is observed critically with a magnifier, the
deterioration of definition away from the best focus is seen much
more easily, and will be apparent when certain differences of "optical
path" arise in the image. This question must be discussed by the
methods of Physical Optics; see Appendix II, and also the material
in Chapter IV.

The limited depth of focus in the image space has, as a necessary
consequence, a correspondingly limited depth of focus, or "focal
depth" in the object space. That is, the object may change its
position within the relevant range of focal depth without appreciable
loss of definition in a fixed image plane. A brief discussion is given
in Appendix III.

Effects of Dispersion

The refractive index has been treated so far as a single-valued
parameter of the medium. However, in actual refracting media the
dispersion of light is one of the most obvious phenomena, and a
necessary preliminary to the discussion of instruments is an account
of the effects of dispersion and the means by which they may be
minimized. Chromatic aberration appears as an aberration even
within the paraxial limits. It occurs in general whenever a ray is
refracted with an appreciable angle of incidence, and two main
effects may be distinguished in the paraxial region—
(i) Chromatic variation in the position of the image plane conjugate to a given object plane.

(ii) Chromatic variation in the size of the image.

These may be illustrated by Fig. 43, showing the (exaggerated) effects of a single surface; the paraxial foci conjugate to B for red rays and blue rays lie at $B_r'$ and $B_b'$ respectively. If the incident light is white, the foci for successive spectral colours are distributed continuously along the axis. Further, considering the principal ray from $B_1$ through the centre of the diaphragm, the off-axis image points are at $B_{1r'}$ and $B_{1b'}$ respectively, for red and blue rays; there will be a continuous variation with incident white light.

Fig. 43.

There will thus be no really sharp physical focus. If yellow rays from B focus at one point there will be circular patches of appreciable size in that focal plane for rays of other spectral colours. Moreover if the principal ray is pictured as the centre of a group of rays it will be realized that off-axis image points are affected not only by this difference in the image-plane position, but also by the difference in the heights, above the axis, of the centres of the coloured image patches.

The special case of a thin lens is worth mention. The axial chromatic difference is encountered as before, but if, as is sometimes the case, the principal ray traverses the centre of the lens, then the deviation for any wavelength (equation (1.28)) is inappreciable. Remember that the tangents to the surfaces at the poles are parallel planes. Therefore there will be no appreciable chromatic difference of magnification unless the diaphragm does not coincide with the lens so that the principal ray is deviated in its course.

**Correction of Dispersion**

It is found that a prism of flint glass produces more dispersion for a given deviation than one of crown glass. Thus if a prism of crown (base down) is combined with one of flint (base up) the deviations
produced tend to annul one another; but owing to the above property the weaker dispersion of the crown can be compensated by the use of a suitable prism angle for the flint, with considerable deviation still left. The combination constitutes an "achromatic prism." A similar compensation occurs in the achromatic lens, shown in Fig. 44. Of course the general discussion of achromatic combinations covers all transparent media, not merely the glasses.

Inaccurate measurements of refractive indices (almost inevitable before the discovery of spectrum lines) had led Newton to believe that dispersion in all media was proportional to deviation, and that achromatic refracting systems were therefore impossible. Fortunately, however, this view was later shown to be mistaken.

**Dispersion Measurements**

The dispersion of media is usually measured for optical purposes in terms of their refractive indices for frequencies of light corresponding to a number of conventionally used spectrum lines. This practice originated with Fraunhofer, who employed the dark lines of the visible solar spectrum, called by him A, B, C, D, etc. In modern times bright lines of certain emission spectra are denoted as follows, some being identical in wavelength with the Fraunhofer lines.

<table>
<thead>
<tr>
<th>Wavelength (Ångström Units)</th>
<th>Denotation</th>
<th>Spectrum of</th>
</tr>
</thead>
<tbody>
<tr>
<td>4047</td>
<td>h</td>
<td>Mercury</td>
</tr>
<tr>
<td>4341</td>
<td>G'</td>
<td>Hydrogen</td>
</tr>
<tr>
<td>4359</td>
<td>g</td>
<td>Mercury</td>
</tr>
<tr>
<td>4861</td>
<td>F</td>
<td>Hydrogen</td>
</tr>
<tr>
<td>5461</td>
<td>e</td>
<td>Mercury</td>
</tr>
<tr>
<td>5875</td>
<td>d</td>
<td>Helium</td>
</tr>
<tr>
<td>5893</td>
<td>D</td>
<td>Sodium (mean of D₁ and D₂)</td>
</tr>
<tr>
<td>6563</td>
<td>C</td>
<td>Hydrogen</td>
</tr>
<tr>
<td>7065</td>
<td>b</td>
<td>Helium</td>
</tr>
</tbody>
</table>

Lists giving the refractive indices of optical glasses for all these wavelengths are usually published by the makers.
In early days, the only glasses known were crown glass (approximately 72% silica, 18% potassium oxide, 10% calcium oxide) and flint (approximately 45% silica, 12% potassium oxide, 43% lead oxide). It was found, however, that by the addition of various substances, i.e. boron, barium, phosphorus, zinc, etc., considerable variations not only in the ratio of the total dispersion to the deviating power, but also in the "run" \((d\mu/d\lambda)\) of dispersion in parts of the spectrum, could be effected. In more recent times various isotropic crystals, e.g. lithium fluoride, potassium bromide, etc., have been used for optical systems, and also various plastics have been found useful (notably "Perspex" and "Polystyrene"). The reasons for the interest in large numbers of materials will become more clear from considerations given below, although the complete discussion is beyond the scope of this book.

**Achromatic Thin Lens Doublet**

If the components of a thin lens doublet with negligible separation have powers \(F_1\) and \(F_2\), the total power \(F\) is

\[
F = F_1 + F_2
\]

but it is essential to write down the sum for some specified wavelength; and the dependence of power on wavelength is implied in each case by such an equation as

\[
F_1 = (n - 1)R_1
\]

where \(R_1\) is the "total curvature" (see equation (1.26)).* Using an additional subscript to denote the wavelength we shall then obtain for the C and F wavelengths respectively—

\[
F_C = (n_{1C} - 1)R_1 + (n_{2C} - 1)R_2 . \quad (3.01)
\]

\[
F_F = (n_{1F} - 1)R_1 + (n_{2F} - 1)R_2 . \quad (3.02)
\]

Now in the case of the "thin lenses" considered, since the focal length of the combination is the reciprocal of the total power, we shall obtain the same focal lengths for the wavelengths C and F if we equalize the corresponding powers. The condition that this may be possible is found by putting \(F_C = F_F\), thus obtaining

\[
(n_{1F} - n_{1C})R_1 + (n_{2F} - n_{2C})R_2 = 0
\]

or

\[
\frac{R_1}{R_2} = \frac{n_{2F} - n_{2C}}{n_{1F} - n_{1C}} = -\frac{\delta n_2}{\delta n_1}
\]

where \(\delta n\) is written for \(n_F - n_C\).

* The suffixes 1 and 2 will now be used in this section to denote lenses, not surfaces as before.
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Since \( n_F \), the refractive index for the shorter wavelength, is for all glasses higher than \( n_C \), \( R_1 \) and \( R_2 \) must have opposite signs. We can reintroduce the powers of the lenses into the equation (i.e. the power for some specified wavelength such as d) by substituting

\[
R_1 = \frac{F_1}{n_{1d} - 1}
\]

and similarly for \( R_2 \)

Thus

\[
\frac{F_{1d}}{F_{2d}} = -\frac{n_{1d} - 1}{\delta n_1} \cdot \frac{\delta n_2}{n_{2d} - 1}
\]  
(3.03)

The ratio \( V \) for any glass is defined by

\[
V = \frac{n_d - 1}{\delta n}
\]  
(3.04)

and is called the "\( V \)-number" or "constringence"* of the material; it is obviously the ratio of the angular deviation for a ray of wavelength d to the difference of angular deviation for rays of wavelengths C and F which would be produced by a very thin prism. Hence

\[
\frac{F_{1d}}{F_{2d}} = -\frac{V_1}{V_2}
\]  
(3.05)

and thus

\[
\frac{F_{1d} + F_{2d}}{F_{2d}} = \frac{F_d}{F_{2d}} = \frac{V_2 - V_1}{V_2}
\]

We thus obtain the necessary powers of the doublet members if the total power is to be \( F_d \)

\[
F_{1d} = F_d \frac{V_1}{V_1 - V_2}
\]  
(3.06a)

\[
F_{2d} = F_d \frac{V_2}{V_2 - V_1}
\]  
(3.06b)

and the corresponding relations between the focal lengths can be found by inverting these last equations, thus—

\[
f_{1d}' = f_d' \frac{V_1 - V_2}{V_1}
\]  
(3.07a)

\[
f_{2d}' = f_d' \frac{V_2 - V_1}{V_2}
\]  
(3.07b)

The \( V \)-numbers of glasses given in accordance with equation (3.04)

* The reciprocal of the \( V \)-number is often called the "dispersive power," but it is not listed in actual tables of optical glasses.
are usually tabulated in lists of optical glasses. A brief list is given in Appendix IV, p. 202.

**Numerical Example.** A telescope objective of focal length 1 metre is to be made from the following glasses—

<table>
<thead>
<tr>
<th></th>
<th>(n_d)</th>
<th>(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light barium crown</td>
<td>1.5406</td>
<td>59.5</td>
</tr>
<tr>
<td>Extra dense flint</td>
<td>1.6511</td>
<td>33.5</td>
</tr>
</tbody>
</table>

Then from (3.07a) \(f_{1d}' = 1 \times \frac{26.0}{59.5} = 0.437\) m

and from (3.07b) \(f_{2d}' = 1 \times \frac{-26.0}{33.5} = -0.776\) m

The powers of these two lenses are thus 2.29 D and \(-1.29\) D respectively, the sum of which is the 1 diopter required. The shapes of the lenses have, however, to be chosen so as to avoid the monochromatic aberrations (see Chapter VII).

**The Secondary Spectrum**

It will be realized that while the "correction" given in the above manner equalizes the focal lengths for the C and F lines, it is not necessarily true that the chromatic aberration has been completely removed, and it is therefore necessary to ascertain how the foci for other components of white light are distributed.

By analogy with equations (3.01) and (3.02),

\[
\frac{1}{f_d'} = (n_{1d} - 1)R_1 + (n_{2d} - 1)R_2
\]

\[
\frac{1}{f_F'} = (n_{1F} - 1)R_1 + (n_{2F} - 1)R_2
\]

Subtracting

\[
\frac{1}{f_F'} - \frac{1}{f_d'} = \frac{f_d' - f_F'}{f_d' f_F'} = (n_{1F} - n_{1d})R_1 + (n_{2F} - n_{2d})R_2
\]

or

\[
\frac{f_d' - f_F'}{f_F'} = f_d' (n_{1F} - n_{1d})R_1 + f_d' (n_{2F} - n_{2d})R_2
\]

But using equations (3.07) above,

\[
\frac{f_d' - f_F'}{f_F'} = (n_{1F} - n_{1d})R_1 f_{1d}' \cdot \frac{V_1}{V_1 - V_2} + (n_{2F} - n_{2d})R_2 f_{2d}' \frac{V_2}{V_2 - V_1}.
\]  \(\text{(3.08)}\)
But \( R_1 V_1 = \frac{1}{f_{1d'} (n_{1d} - 1)} \times \frac{n_{1d} - 1}{\delta n_1} = \frac{1}{f_{1d'} \delta n_1} \)
and a similar equation can be written for \( R_2 V_2 \); thus, substituting in (3.08) it is found that
\[
\frac{f_{d'} - f_{F'}}{f_{F'}} = \frac{n_{1F} - n_{1d}}{\delta n_1 (V_1 - V_2)} + \frac{n_{2F} - n_{2d}}{\delta n_2 (V_2 - V_1)} .
\]
(3.09)
from which the difference between the focal lengths for the d and F wavelengths can be calculated as a fraction for the combined focal length for F. Now the figures for a number of the partial dispersions such as \( n_F - n_d \) are ascertainable from the tables (see Appendix III) and they are also usually quoted as fractions of the dispersion \( \delta n \) (i.e. the difference \( n_F - n_C \)), so that it is easy to see almost immediately from the tables how the differences of the focal lengths for lines other than C and F are distributed. Let us put
\[
\alpha = \frac{n_d - n_o}{\delta n}, \quad \beta = \frac{n_F - n_d}{\delta n}, \quad \gamma = \frac{n_o - n_F}{\delta n}
\]
The equation (3.09) can then be written
\[
\frac{f_{d'} - f_{F'}}{f_{F'}} = \frac{\beta_1 - \beta_2}{V_1 - V_2} \quad . \quad . \quad . \quad (3.10)
\]
and parallel methods will yield
\[
\frac{f_{b'} - f_{d'}}{f_{b'}} = \frac{\alpha_1 - \alpha_2}{V_1 - V_2} \quad . \quad . \quad . \quad (3.11)
\]
\[
\frac{f_{F'} - f_{g'}}{f_{g'}} = \frac{\gamma_1 - \gamma_2}{V_1 - V_2} \quad . \quad . \quad . \quad (3.12)
\]
**Numerical Example.** Extracting from a table the data for glasses crown and flint, suppose we find
\[
\frac{f_{b'} - f_{d'}}{f_{b'}} = \frac{0.489 - 0.457}{60.3 - 37.8} = 0.032 \quad \quad 22.5
\]
\[
\frac{f_{d'} - f_{F'}}{f_{d'}} = -0.01 \quad \quad 22.5
\]
\[
\frac{f_{F'} - f_{g'}}{f_{g'}} = -0.035 \quad \quad 22.5
\]
These are all small fractions; therefore all the focal lengths are so nearly equal that we may regard the equation as expressing the ratio of the differences to the general focal length; and the three differences above have the ratio 2.2, -1.0, -3.5. Remembering
that $f_F'$ and $f_C'$ are equal by the assumed construction of the lens, we may draw the curve in Fig. 45, showing the relative variation of the focal point position with wavelength.

It is easy to show that the $V$-number expresses the ratio of the total focal length of a single thin lens to the interval between its principal foci for $C$ and $F$. Thus the extent of the axial spectrum so measured is, for a crown lens, about $\frac{1}{10}$th of its focal length. In the "achromatic" lens the spectrum is seen to be "folded" along the axis with a minimum focal length not far from the yellow (d) focus,

![Fig. 45. The secondary spectrum.](image)

and the interval between this and the combined $C$ or $F$ focus is, from the figures given above, $\frac{1}{2.56}$th of the focal length. The spectrum is therefore greatly reduced, though it may still set a limit to the relative aperture which may safely be given to an objective.

The abolition of the secondary spectrum calls for pairs of glasses with identity of all the relative partial dispersions. These, however, are not available in technically satisfactory glasses, and some amount of secondary spectrum has usually to be tolerated. Equations (3.10) to (3.12) show, however, that the trouble may generally be reduced by using pairs of glasses with the maximum possible difference between the $V$-numbers; this has the added consequence (as will be appreciated from equations (3.07)) that the powers of the component lenses will tend to be small and the curvatures therefore relatively modest; this in turn tends towards small angles of incidence for the rays refracted at the surfaces and thus to relatively small monochromatic aberrations.
The introduction of fluorite crystal into a lens system is sometimes useful, as it has a run of relative partial dispersions which is very similar to those of certain glasses, and it has a very high V-number (95-4). It is thus possible to obtain very high degrees of correction of both chromatic and monochromatic aberrations by its use. It can only be obtained in relatively small pieces and is mainly used in microscope objectives. Other substances can sometimes be employed in a similar way.

Some notes on dispersion formulae are given in Appendix V.

**Fig. 46.**

**Triplet Telescope Objectives**

A closer approach to perfect chromatic correction than is possible with a doublet construction was obtained by Dennis Taylor. He used three lenses; two, of approximately equal positive powers, were made of glasses such that the means of their relative partial dispersions were approximately equal to those of the borosilicate flint used for the accompanying negative lens.

**The Magnifying Glass or Simple Microscope**

In order to obtain the most advantageous view of a small object, it is brought as near to the unaided eye as accommodation will allow. The variation of accommodation with age, etc., will be briefly described in Chapter V, but it can be assumed that there will be some distance $b^*$ which, measured from the nodal point of the eye, marks the closest possible approach of a real object consistent with sharp vision. If the object itself of height $h$ is held at this distance, the

* Conforming to Fig. 46, $b$ will be treated as numerically negative.
angle \( \omega \) under which it is seen (assuming paraxial conditions as defined in Chapter I) is (Fig. 46)

\[
\omega = - \frac{h}{b}
\]

This angular subtense determines the size of the retinal image. If, however, a positive lens is held before the eye, the object may be brought still nearer, until the image is now situated at the distance \( b \). In a case where the lens (of power \( F \)) is thin, and where the distance between lens and eye may be neglected, the distance \( l \) of the object from the lens must conform to the equation.

\[
\frac{1}{b} - \frac{1}{l} = F
\]

from which

\[
l = \frac{b}{1 - Fb}
\]

The angle \( \omega' \) under which the image is now viewed is

\[
\omega' = - \frac{h}{l}
\]

and if the angular magnifying power \( M \) is given by

\[
M = \frac{\text{angle under which image is seen with optical aid}}{\text{angle under which image is seen without optical aid}}
\]

then

\[
M = \frac{-h(1 - Fb)b}{-h/b} = 1 - Fb \quad \text{(3.13)}
\]

If the magnifier is used with relaxed accommodation of the (normal) viewing eye, the object is held at the front principal focus of the lens in order to obtain an image at infinity. The angular subtense of the image, supposing the principal ray to pass through the centre of the lens, is then

\[
\omega' = - \frac{h}{f} = hF
\]

so that the magnifying power in such a case is

\[
M = \frac{\omega'}{\omega} = \frac{hF}{-h/b} = - Fb \quad \text{(3.14)}
\]

**Numerical Example.** If the power \( F \) is given in diopters, \( b \) must be expressed in metres; thus if the minimum distance of distinct vision is 25 cm, \( b = -0.25 \) m. So that if the lens is of power +10 D, the magnifying power is 2.5 if the lens is used with the unaccommodated, and 3.5 if with the accommodated eye.

**Telescopes and Compound Microscopes**

The basic principle of both these instruments is that a real image of an object is projected by a suitable lens or optical system (the
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objective) into a convenient position for observation, usually with the aid of a magnifying system (the eyepiece). A "telescope" could, in principle, be constructed without any lenses; for example an image of the sun (Fig. 47) could be thrown on a screen in a dark room with the aid of a pinhole in a shutter. An observer in the room could approach the image as closely as he pleased (consistent with sharp vision) and thus see the sun's image under a larger angle than with direct observation. If the angle subtended by the sun at the pinhole is \( \omega \) and the distance from pinhole to screen is \( l_1 \), the diameter of the image on the screen is approximately \( \omega l_1 \). If the distance of the observing eye from the screen is \( l_2 \) (and the direction of observation is close to the normal) the angular size of the image is \( \omega l_1/l_2 \).

Hence \[ \text{magnifying power} = \frac{\omega l_1}{l_2} = \frac{l_1}{l_2} \]

The Astronomical Telescope ; Paraxial Theory

By analogy with the above, the objective or object glass of the astronomical telescope (Fig. 48) projects an image of an infinitely distant object into its focal plane at a distance \( f_0' \) from the second principal plane of the lens. If the angular subtense of the object is
\( \omega \), the size of the image is \( \omega f_0 \); this is to be viewed by a magnifier of second focal length \( f'_e \). The instrument is adjusted so that the first principal focus of the magnifier coincides with the internal image, when the final image will be at infinity and its angular subtense at the lens will be \( (\omega f_0) f'_e \); this is the angle \( \omega' \) under which it is seen by the observing eye.

Now if the object is at infinity, it subtends the same angle \( \omega \) from any possible viewpoint (and the same applies to the final image), so that the angular magnifying power of the instrument is given by

\[
M = \frac{\omega'}{\omega} = \frac{\omega f_0 f'_e}{\omega} = f_0 f'_e . \tag{3.15}
\]

This discussion has been so phrased that it will apply to a telescope in which the internal medium is not air. If, however, the objective and eyepiece are both lenses in air, then we may replace \( f_0 \) in the equation above by \(- f'_e \), so that the magnifying power is the negative quotient of the ordinary focal length of the objective divided by that of the eyepiece—or alternatively of the power of the eyepiece divided by the power of the objective.

\[
M = -\frac{F_e}{F_0} . \tag{3.16}
\]

The negative sign indicates the inversion of the image if both \( F_e \) and \( F_0 \) are positive.

The Gaussian properties of the system as a whole are merely of academic interest (see Exercises 4 and 12, p. 105), but the ray diagrams showing the course of rays reaching the viewing eye should be drawn by the student after due consideration of the position of the eye which permits the maximum possible angular field of view.

In the case of a thin objective in a simple mount, used with an eyepiece of adequate aperture, the entrance pupil of the system is defined by the circular diaphragm limiting the objective, since all the parallel ray paths derived from an infinitely distant axial object point will pass unobstructed through the system—and will be parallel on emergence (Fig. 49). In accordance with the definitions on p. 71, the exit pupil \( B'B'_e \) will then be the image of the objective aperture \( BB_e \) formed by the remainder of the system, i.e. the eyepiece. The exit pupil must thus transmit all the ray paths which can reach the eyepiece from the objective, and is, therefore, the correct position for the entrance pupil of the observing eye; if the aperture of the eye is large enough, all the same ray paths will reach the retina and thus secure the maximum possible illumination of the image.
The rays shown in Fig. 49 will illustrate the foregoing conclusions. Consideration of the ray path ABCDB' will at once show that the ratio of the diameters of the entrance and exit pupils is numerically equal to the ratio of the two internal focal lengths, i.e. to the magnifying power of the system. This conclusion is in accordance with the Smith–Helmholtz relation (1.40), i.e.

\[ nh_u = n' h' u' \]

If this equation is applied to the pupils of a telescope in air, the ratio \( u'/u \) is the angular magnifying power; since \( n \) and \( n' \) are unity the ratio is equal to \( h/h' \), i.e. the linear magnification ratio for the (conjugate) pupil positions.

Field of View

The field of view may be limited by a diaphragm placed in the common focal plane where it directly controls the size of the image transmissible by the instrument. The images of this diaphragm formed by the objective and eyepiece respectively are sometimes termed the "windows" of the system. If, however, the eye is moved axially away from the exit pupil, it may not receive some of the oblique bundles of rays, and the exit window is not seen fully illuminated. In most telescopes there is an "eye-ring" or some arrangement so that it is easy to keep the correct position for the eye when the eye-ring rests against the bony structure of the periphery of the orbit.

Chromatic Correction

Considering the rays shown in Fig. 50, it is evident (as explained above, p. 38) that while the principal ray passes through the centre
of a comparatively thin objective lens (or lens-pair) there will be no lateral chromatic aberration in the intermediate image; although the axial chromatism makes a single lens unsuitable, and it is accordingly replaced by an achromatic system.

When (as in Fig. 49) the exit pupil does not coincide with an eyepiece lens a principal ray is transmitted eccentrically by such a lens, and lateral chromatic aberration will therefore appear. It is thus necessary to use a corrected eyepiece system. A cemented achromatic lens is sometimes used as an eyepiece in an astronomical telescope. It was found, however, notably by Huygens, that correction could be obtained by the use of two separated single lenses. The principal ray from the centre of the objective (see also Fig. 51) is dispersed by the first lens of the pair (the field lens), and accordingly the blue ray meets the second lens (the eye lens) nearer to the axis than the red ray. The net deflexion at the second lens being thus greater for the red ray than for the blue, the paths of the red and blue principal rays can be made parallel on emergence. Since, however, the principal rays are the central rays of their respective parallel bundles of red and blue rays, the red and blue images, at infinity, will subtend the same angle in the field of view seen by the eye.

Recalling the definition of the focal length of the image space given on p. 52, and remembering that with an achromatic objective the intermediate red and blue images will be of equal size, it will be understood that the condition to be fulfilled by the eyepiece combination is that its focal lengths for red and blue rays respectively
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should be equal, or alternatively the powers should be equal. This can be achieved even if the two lenses are made of the same glass. We shall discuss it as a thin-lens problem.

Let us first consider the condition which must be satisfied if a system of two thin separated single lenses is to give images of equal size in two colours.

For a system in air, the Smith–Helmholtz equation gives \( m = u_1/u_2' \) where \( m \) is the magnification \( h_2'/h_1 \). Therefore if \( u_1 \) remains constant while the wavelength changes, a constant magnification requires that \( u_2' \) must remain unchanged, that is

\[
\frac{\delta u_2'}{\delta \lambda} = 0
\]

An expression for \( u_2' \) can readily be written by taking the analogue of equation (2.22) for the case of two separated thin lenses, the mutual distance being \( a \), i.e.

\[
u_2' = y_1(F_1 + F_2 - a F_1 F_2) + u_1(1 - F_2 a)
\]

Assuming that the distance of an axial object from the first lens is \( l_1 \), the equation becomes

\[
u_2' = y_1((F_1 + F_2 - a F_1 F_2) + (1 - F_2 a)/l_1) \quad (3.16)
\]

If the wavelength of the incident ray changes without a change in \( y_1 \) or \( l_1 \), the rate of change of \( u_2' \) will be found by differentiating, thus

\[
\frac{\delta u_2'}{\delta \lambda} = y_1 \left( \frac{dF_1}{d\lambda} + \frac{dF_2}{d\lambda} - a \left( \frac{dF_1}{d\lambda} F_2 + \frac{dF_2}{d\lambda} F_1 \right) \right)
\]

Now since for a thin lens \( F = (n - 1)R \), where \( R \) is the total curvature,

\[
\frac{dF}{d\lambda} = \frac{dn}{d\lambda} R
\]

Provided that the change \( \delta n \) is very small, such equations can be used without appreciable error to discuss the effect of a finite change of wavelength \( \delta \lambda \), as from \( C \) to \( F \), and we can thus write—

\[
\frac{\delta F}{\delta \lambda} \approx \frac{\delta n}{\delta \lambda} R
\]

Introducing the \( V \)-number for the glass, we have

\[
\frac{\delta F}{\delta \lambda} \approx \frac{1}{\delta \lambda} \left( \frac{dn}{n_d - 1} \right) \left( n_d - 1 \right) R = \frac{1}{\delta \lambda} \cdot \frac{F}{V} \quad (3.18)
\]

where \( F \) is the power of the lens for the d line.
The foregoing equation (3.17) can now be written

$$\frac{\delta u_2'}{\delta \lambda} \approx \frac{y_1}{\delta \lambda} \left[ \frac{F_1}{V_1} + \frac{F_2}{V_2} - \alpha F_1 F_2 \left( \frac{1}{V_1} + \frac{1}{V_2} \right) - \frac{a}{l_1 V_2} \right]$$

If $y_1$ is finite, the condition that the system may be achromatic with regard to the size of the image is thus that the value of the square bracket on the right of the above equation may be zero; i.e.

$$\frac{F_1}{V_1} + \frac{F_2}{V_2} - \alpha F_1 F_2 \left( \frac{1}{V_1} + \frac{1}{V_2} \right) - \frac{a}{l_1 V_2} = 0 \quad (3.19)$$

If $l_1$ is infinite, the last term on the left of the last expression vanishes; then, introducing the focal lengths of the lenses, we obtain the convenient result, for the important case where the two lenses are of the same glass so that $V_1 = V_2$:

$$a = \frac{f_1' + f_2'}{2} \quad \quad \quad (3.20)$$

This result is often sufficiently near in cases where the eyepiece is to be used with a telescope objective of relatively great focal length. Note that condition (3.19) reduces to the ordinary condition for the achromatism of a close doublet when $a = 0$.

The fulfilment of these conditions does not prevent the appearance of axial chromatic aberration (suggested by the interval $\delta \lambda$ in Fig. 51).

Having regard to the requirements of the Smith–Helmholtz relation, it will be clear that the condition for the freedom of a system of two separated thin lenses from both axial and lateral chromatism is that the two relevant coloured rays from an axial object point must coincide in space after emergence from the second lens. If the latter is of negligible thickness, the same rays must coincide in space between the lenses. Hence the condition above can only be fulfilled if both the thin lenses are separately achromatic, i.e. they might both be thin doublets. This conclusion does not apply to lenses of considerable thickness.

Although a system of two separated thin lenses of the same glass can be freed from lateral colour, it will still possess axial chromatic aberration in the sense that if the rays derived from an axial object point are to be parallel to the axis on emergence, the point must be closer to the first lens for blue than for red rays. Eyepieces of such a kind are therefore unsuitable for use with objectives which have a common focus for red and blue. However, since it is the action of the flint lens in a doublet objective which tends to lengthen the focal length for the blue relatively to the red, it is possible to design the
objective with a somewhat increased power of the flint, so that it is so far "over-corrected" that the red and blue foci lie in the positions required by the eyepiece. Then the instrument is nearly corrected as a whole so far as the two paraxial types of chromatic error are concerned.

**Properties of Eyepieces**

A typical Huygens eyepiece is shown in Fig. 52(a), together with the positions of its principal and focal points in monochromatic light. It will be seen that the first principal focal plane, into which the intermediate image must be projected by the objective, lies between the lenses. The focal length of the field lens is usually twice

or more that of the eye lens, but seldom as much as thrice. The separation of the adjacent principal points of the two lenses is, in accordance with equation (3.20), approximately the mean of the two focal lengths.

The Ramsden eyepiece (Fig. 52(b)) was designed because it is of some importance to be able to measure directly the image formed by the objective, e.g. by the use of "cross threads" (sometimes made of spider web) moved in the focal plane by the means of a measuring micrometer screw. (The interposition of the field lens of an eyepiece such as the Huygenian might cause distortion.) Ramsden used two lenses of equal focal lengths. According to the rule above, the separation would be equal to the common focal length, which would, however, mean that one of the lenses would coincide with the first principal focus, where the image has to be formed. In order to allow the cross-threads room to operate and also to keep the exit pupil at a reasonable distance behind the eye lens, the lenses are therefore mounted closer together, so that the focal plane lies at a sufficient distance from the lens surface. This entails some departure from complete lateral achromatism, but it can usually be tolerated if the
telescope has not a very wide field of view. In Ramsden's day telescopes were usually unsealed, and liable to accumulate dust and moisture on the interior surfaces, so that this was an additional reason for preferring to keep any such surface away from the focal plane, where the dirt might be clearly seen when using the instrument. In modern times dust, etc., is carefully excluded and glass plates bearing scales (known as graticules) are often mounted in the focal planes for measuring the image as in binoculars or "sighting telescopes."

The imperfection of the colour correction in a Ramsden-type eyepiece can be remedied if the eye-lens is made a compound system as in the "Achromatic Ramsden" type. A great variety of eyepieces is now made, and the aims of their design will be more clearly understood when some account of aberrations (Chapter VII) has been given.

**Erecting Systems using Lenses**

The earliest erecting system for telescopes employed an auxiliary lens system intermediate between the objective and eyepiece proper, so that the first intermediate inverted image was projected erect for examination. The relative conjugate distances were variable by the operation of the draw-tubes, so that a variable magnification was obtained. A similar system, with properly designed elements, is used in modern "variable-power" telescopes.

A system using an erector formed by two separated single lenses was designed by Fraunhofer (Fig. 53); systems of this kind are still used in coastguard telescopes and the like. The axial chromatic aberration of the erector and eyepiece system taken together is severe, so that the objective used for such telescopes has to be specially designed and is not suitable for use with astronomical eyepieces; the lateral colour can, however, be well corrected.

**Prismatic Erecting Devices**

Reflection in a plane horizontal mirror (Fig. 54) produces *inversion*, and reflection in a vertical mirror produces *reversion* (left to right)
of an image. These are necessary consequences of the laws of reflection, and the images are otherwise free from any aberration. It follows that if rays are successively reflected at a horizontal and a vertical plane surface, or vice versa, the image is completely inverted and reversed. The order of reflection has no effect on the position of the final image; hence the images formed by rays reflected in both ways are perfectly coincident if the surfaces are truly perpendicular. In Fig. 55 (a) and (b), the third image is formed by rays which have suffered two reflections by mutually perpendicular mirrors; and it immediately follows from the geometry of the case that the conjugate image is found for any point of a three-dimensioned object by dropping a perpendicular to the line of intersection of the two mirrors and marking off an equal distance beyond this line. This being the case, the image doubly-reflected remains stationary even if the mirror system rotates as a whole about the line of intersection as an axis (Fig. 55(b)).

It follows that such double reflections of the rays proceeding towards an inverted image will completely erect it (see Fig. 56); but if the mirrors are small, the bundle of rays from an objective must fall on them at a fairly oblique angle in order that all may be reflected, and a change of direction is inevitable. Sometimes the
reflecting surfaces are formed on the faces of a prism of glass (Fig. 57); the intersection edge is called a "roof" reflector. The glass faces of entrance and exit of the light have to be perpendicular to the optical axis of symmetry of the dioptric elements; this axis thus suffers a change of direction at its meeting point with the roof. The optical action of the whole prism is thus equivalent to that of the reflecting mirrors plus a glass plate with plane faces perpendicular to the axis.

The final direction of the axial ray may be made parallel to its initial direction by a system invented by Porro. Two right-angled prisms are used, each with the transmitting face parallel to the line of intersection of the perpendicular reflecting surfaces (Fig. 58). The "line of intersection" of the second prism should be strictly perpendicular to that of the first. Each prism is then equivalent to the pair of reflectors plus a plate of glass which need not enter into consideration in the erecting effect. Suppose that the action of the first prism swings the image 180° about one axis (say OZ, Fig. 59) which brings it into the first position (a); then the second prism

![Perspective view](image1) ![Side view](image2)

**Fig. 57. Erecting prism.**

swings it about OX into the position (b) where it is inverted and reversed with respect to its original orientation. The final direction of a ray is parallel to its original direction though it suffers a lateral shift.

The roof reflectors of the former type suffer from the danger that if the angle between the reflecting faces (a and b, say) is not exactly 90°, the position of the image will differ slightly according to the order in which the rays are reflected, i.e. whether first at a, and then at b, or vice versa. The angular separation of these images due to the reflection alone, and consequently overlapping with regard to
their subtense at the roof line, can be shown to be four times the difference from 90° of the angle between the faces. The manufacture of the prisms has therefore usually to be meticulously accurate if a visible doubling is to be avoided. On the other hand it will be evident that in the Porro system the successive surfaces are encountered by all the rays in the same order, and the only considerable effects of any defect in the accuracy in the angles of the prisms or their adjustment will usually be in the apparent direction or orientation of the image.

![Porro prisms](image1)

**Fig. 58.** Porro prisms.

![Orientation and position of eyepiece](image2)

**Fig. 59.**

Numerous other types of prism erecting systems are described in technical works.

**Galileo’s Telescope**

Although telescopes with eyepieces of negative power are generally called “Galilean,” the primitive invention seems to have been shared in some way unknown between the Dutch spectacle-makers Jansen and Lippershey of Middleburg about 1609. Galileo, however, developed the instrument in Florence and was the first to apply it to astronomical observations. The use of a positive eyepiece for the construction of an astronomical glass by Kepler soon followed.

Equation (3.16) shows that if $F_e$ and $F_o$ have opposite signs the image will be erect. The Gaussian diagram is shown in Fig. 60(a), while typical ray paths are illustrated in Fig. 60(b). There is, of course, no real intermediate image, since the rays are parallelized by the negative eyepiece before they reach a focus; both objective and eyepiece are usually compound achromatic cemented lenses.

Since there is no real image, there can be no field-stop in the focal plane; moreover the pupil of the eye has to be situated at some reasonable distance behind the eye-lens. The exit pupil, which in
this case is the image of the objective aperture formed by the eyep- 

e lens, is situated between the lenses, and the eye cannot be placed 

there; accordingly in this case the exit pupil acts as the exit window 

controls the visible field of view.

The angular subtense of the exit pupil at the eye position N (Fig. 60(b)) at a distance \( d \) behind the eye-lens can thus be calculated 

for the thin lens case. The distance of the objective from the eye-

lens is \(- (f'_1 + f'_2)\), where \( f'_1 \) and \( f'_2 \) are the image-space focal 

lengths of objective and eyepiece respectively.

Accordingly the image of the objective is at a distance \( l'_2 \) from the 

eye-lens where

\[
\frac{1}{l'_2} + \frac{1}{f'_1 + f'_2} = \frac{1}{f'_2}
\]

or

\[
l'_2 = \frac{f'_2(f'_1 + f'_2)}{f'_1} = f'_2 \left( 1 - \frac{1}{M} \right)
\]

where \( M = -f'_1/f'_2 \), is the angular magnifying power.

Hence the distance of the eye from the exit pupil is

\[
d = f'_2 \left( 1 - \frac{1}{M} \right)
\]

But since the diameter of the exit pupil (if \( p \) is the diameter of the 

objective) is \( p/M \), its angular subtense, which is the angular field of 

view for an eye of negligible pupil size, is given by

\[
\text{angular field} = \frac{p/M}{d - f'_2 \left( 1 - \frac{1}{M} \right)} = \frac{p}{Md + f'_1 \left( 1 - \frac{1}{M} \right)}.
\]  

(3.21)
The effect of the various parameters on the size of the field can be studied from this expression. It will be noted that, other things being equal, a larger field can be obtained by an increase of the diameter of the objective, or by a diminution in the distance of the eye from the eye-lens, or by the diminution of the magnification.

**Reflecting Telescopes**

The use of reflecting or catoptric systems rather than lenses has the enormous advantage that no chromatic aberration appears. But if it is necessary to observe directly an axial image formed by a concave mirror, the head of the observer obstructs a part, if not all, of the light, and some special device is usually necessary. A possible construction was first definitely suggested by James Gregory in 1663 (Fig. 61). The main paraboloidal* mirror M forms its inverted image at B, and an erect image is then projected to B' by the smaller ellipsoidal concave mirror m through a central perforation in M, to a place convenient for the use of an eyepiece. The axial points B and B' are the foci of m. Referring to the known properties of such ellipsoids (see p. 13) and remembering that a paraboloid is an extreme case of an ellipsoid with one focus at infinity, it is apparent that the formation of the point axial image should be completely free from spherical and chromatic aberration.

An alternative and somewhat more convenient form for astronomical purposes (since the telescope is shorter) was put forward by Guillaume Cassegrain in 1672. The main mirror (Fig. 62) is again parabolic and perforated, but the smaller mirror is now a hyperboloid with the foci at B and B', i.e. the principal focus of M and the final real (inverted) image, respectively. These focal properties of the ellipsoid and hyperboloid can readily be demonstrated.†

---

* Surfaces with axial symmetry are implied throughout the discussion.
GEOMETRICAL OPTICS

There is no clear record of any successful telescope actually constructed by either Gregory or Cassegrain, though their suggestions were taken up later with success.

Newton employed an auxiliary plane reflector (Fig. 63), and exhibited a fairly successful model of his reflecting telescope to the Royal Society in 1672. The main mirror was spherical and worked at f/6, being only one inch in diameter. No ready means of working an accurate paraboloid was then known.

The use of the 90° deflexion in the direction of the reflected axial ray is convenient for astronomical purposes, and the device is still frequently used in large telescopes. The method used by Herschel, in which the mirror is tilted away from the normal with respect to the direction of observation requires an unsymmetrical figuring of the surface, and is little used in modern times.

The power of two-mirror reflectors may be readily calculated by the general formula (2.24), applying the rules of p. 68. We had

\[ F = F_1 + F_2 - F_1F_2d \]

The general expression for the power of a surface is \( F = (n' - n)/r \).
For the first mirror, in Figs. 61 and 62, \( n' = -1, n = 1 \) and the power
\[
F_1 = -\frac{2}{r_1}
\]
For the second mirror \( n' = 1, n = -1 \), and the power
\[
F_2 = 2/r_2
\]
If \( d \) is the actual distance of the second mirror from the first, then it has a negative sign, but \( d = d/n' = -d \), so that \( d \) is numerically equal to the separation and has a positive sign. The combined power is readily seen to be
\[
F = \frac{2(r_1 + 2d - r_2)}{r_1r_2} = \frac{1}{f'}
\]  \hspace{1cm} (3.22)

The distance \( A_2F' \) (Fig. 64) of the principal focus of the image space from the second surface in a two-surface system may be readily shown to be
\[
A_2F' = n'(1 - F_1d)/F
\]

\[ \text{Fig. 64.} \]

and in our case the result in terms of the radii, etc., turns out to be
\[
A_2F' = \frac{(r_1 + 2d)r_2}{2(r_1 + 2d - r_2)}
\]  \hspace{1cm} (3.23)

The secondary mirror causes a certain degree of obscuration of the central area; referring to Fig. 64, the innermost parallel ray \( H_2G_1 \) just escapes the edge of the secondary mirror, and is reflected at \( G_1 \) and \( G_2 \); but the outermost ray is reflected at \( H_1 \) and \( H_2 \), hence the proportion \( S \) of the angular aperture which is obscured is
\[
S = \frac{G_2F'A_2/H_2F'A_2}{n'}
\]

Now if the ray \( G_2F' \) is produced backwards until it intersects its incident path, the meeting place is at \( P_1' \) in the second principal plane \( P'P_1' \), where \( P'F' = f' \).

Since \( A_2H_2 = P_1P_1' \), we have
\[
S \simeq \frac{A_2F'}{f'} = \frac{r_1 + 2d}{r_1}
\]  \hspace{1cm} (3.24)
where $d$ is equal (as explained above) to the numerical measure of the separation between the mirrors. This result assumes, of course, that the primary mirror is larger than is necessary for use with the given secondary at the prescribed separation; on the other hand if the primary mirror is too small (or the secondary too large) the obscuration ratio is (for relatively small apertures) closely proportional to the ratio of the apertures of the large and small mirrors.

The use of the paraboloid plus hyperboloid combination has been superseded in more recent times by the use of other aspheric surfaces in Cassegrain systems; the sine condition (p. 125) can then be fulfilled so that better definition can be obtained for off-axis image points. Such combinations have been calculated by Schwarzschild; also independently by Chrétien; and constructed by Ritchey.

**The Microscope**

In the case of the "compound" microscope the object is small and accessible, and the enlarged real image of it projected by the objective is examined by the eyepiece or magnifying system. The Gaussian diagram is shown in Fig. 65.* Let the image-space focal lengths of the objective and eyepiece respectively be $f'_0$ and $f'_e$; their adjacent principal foci are $F'_0$ and $F'_e$, separated by a distance $g$. Let the size of the object BB, perpendicular to the axis be $h$, then the first (real) image projected by the objective will have a size $h'$, where

$$h' = \frac{hg}{f'_0}$$

Since the first image is formed in the anterior principal focal plane of the eyepiece, the rays diverging from the point $B'_1$ (conjugate to $B_1$) in the first image will be parallel after transmission by the eyepiece, and the image is seen under the angle $\omega'$ expressed by the quotient of $h'/f'_e$, i.e.

$$\omega' = \frac{hg}{f'_0f'_e}$$

* The Gaussian properties of such a system are the subject of Exercise 12, p. 106.
However, if the object itself were held at a distance $b$ it would be seen under the angle $\omega$ (see p. 82) where

$$\omega = -\frac{h}{b}$$

Hence

$$\text{magnifying power} = \frac{\omega'}{\omega} = \frac{bg}{f_0 f_e}.$$  \hspace{1cm} (3.25)

The convenient unit of length is the millimetre; then $b = -250$ mm (conventional value) and $g$ is about 160 mm in most modern microscopes. The above formula is valid even if the objective is of the immersion type (see p. 126) as the specified "focal length" is always that of the image space. Thus for example with $f_0' = 2$ mm, and $f_e' = 8$ mm, it is possible to have a magnifying power of 2,500.

The major requirements for the actual construction of the objectives of microscopes will be discussed below (p. 122) in connexion with the theories of resolving power and aberration. For all except the lowest powers, combinations of relatively thick lenses are essential, and this has the consequence that, even if the axial chromatic aberration is corrected, there may be some unavoidable chromatic difference of magnification. Therefore the eyepiece for microscopes may have to be specially designed, and "compensating" eyepieces are used where there is special difficulty of this kind with otherwise highly corrected "apochromatic"* systems. However, in most cases the chromatic difference of magnification is small, and the use of Huygenian eyepieces is common except in cases where larger fields are required.

**Reflecting Microscopes**

Although the use of reflecting microscopes has often been suggested, and various forms of low-aperture systems are made commercially, it was only about 1947 that the difficulties in their construction with high apertures were overcome so far as to secure an improvement over the performance of refracting systems, when C. R. Burch produced a kind of reversed Cassegrain reflector (Fig. 66). The object B, illuminated with the aid of an aspheric condenser C, sends a beam of wide aperture to the main reflector M, whence the rays travel to the small convex mirror m by which they may be brought to the more distant conjugate point. There are special

* "Apochromatic," i.e. distinguished from ordinary achromatic lenses in having a more complete correction for colour, as well as a correction for spherical aberration and coma.
cases in which the system can be corrected for "spherical aberration" (see p. 185) and for "offence against the sine condition" (see p. 125) by the use of one spherical and one aspherical reflecting surface. The system, once focused for visible light, can be used for photography with the ultra-violet, etc.

![Diagram](image)

Fig. 66. Burch reflecting microscope (aspheric surfaces).

**The Prism Spectroscope**

The spectroscope consists in its simple form, Fig. 67, of a collimator, a prism (or other dispersing system), and a telescope (which may be replaced by a camera). The collimator tube carries a diaphragm with a short and narrow slit, the centre of which is placed in the anterior principal focus of the achromatic collimator lens, the length of the slit being perpendicular to the axis. Thus each element of the slit gives rise to an emergent bundle of parallel rays, the principal ray in every bundle lying in the axial plane containing the slit, and all the emergent rays being closely parallel to this plane. If \( h \) is the total height of the slit and \( f \) the focal length of the collimator lens, then \( \pm h/2f \) represents the range of angles between the principal rays and the axis. The bundles of parallel rays now enter a prism adjusted so that its refracting edge (or geometrical line of intersection between the planes of the refracting faces) is parallel to the slit. Fig. 67 shows only the rays belonging to a principal section of
the prism, i.e. a section perpendicular to the refracting edge. Since the angles of incidence and refraction in the prism for all rays of a truly parallel incident bundle will be identical, the corresponding rays emerging from the prism will therefore be parallel also, and will be brought to a sharp focus in the focal plane of the achromatic telescope lens (apart from residual lens-aberrations). The aperture of the beam entering the telescope is commonly delimited by a rectangular diaphragm with edges parallel and perpendicular to the edge of the prism and the slit. There will be a sharp image of the slit for each wavelength of monochromatic light proceeding from it. The dispersion will be discussed in a later section; but first the deviation of the various principal rays in one colour must be discussed, since the position and shape of the slit image is dependent thereon. The first problem is the deviation of a ray lying in a principal section. It is convenient for this purpose to label the angles between the external paths and the normals at the two faces respectively \(i_1\) and \(i_2\), while the internal angles are \(r_1\) and \(r_2\). The total angular deviation \(\Delta\) is

\[
\Delta = (i - r_1) + (i_2 - r_2) \quad (3.26)
\]

all the angles in the diagram (Fig. 68) being taken as positive. However, since it is easily seen that \(r_1 + r_2 = A\), where \(A\) is the angle of the prism,

\[
\Delta = i_1 + i_2 - A \quad (3.27)
\]

Consider the condition when \(i_1 = i_2\), so that the passage of the ray is symmetrical. Then if \(i_1\) increases, \(r_1\) also increases but \(r_2\) decreases and therefore \(i_2\) decreases also; hence \(i_2\) will be less than \(i_1\). From (3.26) above the change of deviation \(\delta\Delta\) may be written

\[
\delta\Delta = \delta(i_1 - r_1) + \delta(i_2 - r_2)
\]

Since \(i_1\) is greater than \(i_2\), it follows from the argument on p. 10 that the positive change \(\delta(i_1 - r_1)\) is greater than the negative change \(\delta(i_2 - r_2)\). Thus the total deviation must increase. Further, since the ray paths are reversible the diminished angle \(i_2\) could be the first angle of incidence, associated with a greater deviation than
that found in the case of equality of \( i_1 \) and \( i_2 \). The deviation is therefore a minimum in the latter case.*

The student should consider what arguments can be used to show that there are no other minima.

The main reason for using the prism in the symmetrical position is that the maximum aperture or breadth of the transmitted beam is then generally available. If the whole apertures of both faces are traversed by rays from a parallel bundle, the prism cannot be turned without restricting the breadth of either the entrant or emergent beam. The effect of this will be discussed in Chapter IV.

It is now necessary to discuss the deviation of rays not contained in a principal section. Consider an isosceles prism of refractive index \( N \) in air (of refractive index unity). The vertical refracting edge is AD (Fig. 69), and BE and CF are the other parallel bounding edges equidistant from AD. The line GP represents the path of a horizontal ray passing in the plane of the base BCFE, but just within the prism and lying in a principal section; its external parts

* The use of a spectrometer permits of the measurement of the angle of the prism \( A \) and the total deviation \( \Delta \). In the case of minimum deviation, \( \tau_1 = \tau_2 = \frac{\Delta}{2} + A \) and \( r_1 = r_2 = \frac{\Delta}{4} A \). Hence the refractive index \( N \) is given by

\[
N = \frac{\sin \frac{\Delta}{4}}{\sin \frac{\Delta}{2}}
\]

Details of the procedure are given in books on practical physics.

It does not appear possible to give a sufficiently simple general expression for \( \Delta \) in terms of \( A \), \( i \), and \( N \) to make an algebraical discussion profitable for the beginner, but advanced students can refer to formulae given by Heath and others. See Heath, Geometrical Optics (Cambridge: University Press, 1895) p. 30.
would then suffer minimum deviation. The line OP is a similar ray in the base but inclined upwards so that GPO is a small angle \( \eta \). Let us discuss the direction of the external paths corresponding to OP.

Heath’s law (p. 11) concerns the projection of the ray paths on a plane containing the normal at the point of refraction; let us take the horizontal plane containing both the normal at P, i.e. Q'PQ, and the line PG. Then if the angle GPQ' is \( \phi \) and QPR is \( \phi' \), where PR is the projection of the ray on the same plane, while \( \eta' \) is the angle between the ray and its projection, Heath’s law (1.07) gives

\[
N \cos \eta \sin \phi = \cos \eta' \sin \phi'
\]

While \( \eta \) is small, its cosine can be expressed very closely by the first two terms in the usual expansion; thus

\[
N \left( 1 - \frac{\eta^2}{2} + O(\eta^4) \right) \sin \phi = \left( 1 - \frac{\eta'^2}{2} + O(\eta^4) \right) \sin \phi'
\]

If we retain no terms of order higher than \( \eta^2 \), we can write

\[
N \left( 1 - \frac{\eta^2}{2} \right) \left( 1 + \frac{\eta'^2}{2} \right) \sin \phi \simeq \sin \phi' \quad . (3.28)
\]

However, equation (1.06) also gives

\[
N \sin \eta = \sin \eta'
\]

Within the order of approximation used above this gives

\[
\eta = \eta' / N
\]

Substituting in (3.28) we thus obtain

\[
N \left( 1 + \frac{\eta'^2}{2} \left( 1 - \frac{1}{N^2} \right) \right) \sin \phi \simeq \sin \phi'
\]

If \( \phi_0' \) is the value of \( i_2 \) (Fig. 68) corresponding to the minimum deviation, the foregoing equation reduces, for the ray in the principal section (\( \eta' = 0 \)), to

\[
N \sin \phi = \sin \phi_0'
\]

Hence a measure of the change of deviation due to the slope is

\[
\sin \phi' - \sin \phi_0' \simeq \frac{1}{2} \eta'^2 \frac{N^2 - 1}{N} \sin \phi
\]

whence

\[
2 \cos \left( \frac{\phi' + \phi_0'}{2} \right) \sin \left( \frac{\phi' - \phi_0'}{2} \right) \simeq \frac{1}{2} \eta'^2 \frac{N^2 - 1}{N} \sin \phi
\]
Thus when $\phi' - \phi_0'$ is very small indeed, the relation leads, within the required approximations, to an expression for the change of horizontal deviation at the one surface—

$$\phi' - \phi_0' \simeq \frac{1}{2} \eta'^2 \frac{N^2 - 1}{N} \sec \phi_0' \sin \phi = \frac{1}{2} \eta'^2 \frac{N^2 - 1}{N^2} \tan \phi_0'$$

(3.29)

Again, consider the implications of Heath's law as applied to a ray for which the internal angle of slope $\eta$ is given. It means that the projected paths obey a modified law of refraction inasmuch as the "apparent refractive indices" are $N \cos \eta$ and $\cos \eta'$, respectively; but the law is otherwise normal. Hence the projected paths will obey the minimum deviation rule for the symmetrical case.

Returning to the result (3.29) above, let us suppose that the ray is traced out into air on both sides of the prism. There will then be a difference of azimuth (horizontal angle) according to the slope, but this is not the case for the incident groups of parallel rays from the collimator, for which the azimuth angle is always the same. However, if we compare a ray from the collimator and the external ray corresponding to the outer path of PO (not shown in the diagram) both having the same slope with respect to the horizontal, their only difference is a slight discrepancy in the value of the azimuth angle. Hence the total horizontal deviation for the ray corresponding to OP and for the collimator ray of the same slope will be sensibly equal. But the total horizontal deviation for the OP rays will be $2\phi' - A$, as compared with $2\phi_0' - A$ for the rays in the principal section. The relative deviation of the sloping rays from the collimator is therefore

$$\delta \phi' = 2(\phi' - \phi_0') = \eta'^2 \frac{N^2 - 1}{N^2} \tan \phi_0'$$

(3.30)

If the telescope lens of the instrument has a focal length of $f'$ and the minimum deviation ray for the principal section is parallel to the optical axis, the image of the (vertical) slit will be a curved line tangential to the vertical direction in the centre of the field. A vertical intercept from the centre being $f'\eta'$, the corresponding horizontal intercept (Fig. 70) will be

$$f'\eta'^2((N^2 - 1)/N^2) \tan \phi_0'$$
whence the radius of curvature is approximately
\[
\frac{1}{2} f' \left( \frac{N^2}{(N^2 - 1)} \right) \cot \phi_0'
\]
This relates, of course, to the condition of minimum deviation only.

The curvature of the lines has to be remembered in accurate spectroscopy and refractometry. The ordinary spectroscope often uses a slit so short that the curvature of the image may escape notice. The various images of the slit formed in all the different frequencies of the light illuminating the slit constitute the spectrum viewed by the eyepiece. If the width of the slit is so great that its image corresponds to an appreciable range of a continuous spectrum characteristic of an infinitely narrow one, the spectrum is said to be "impure." The resolving power of spectrosopes is discussed in Chapter IV.

**EXERCISES III**

1. The exit pupil of a photographic lens (in this case coincident with the second principal plane) has a diameter of 1 in., and the focal length of the lens is 6 in. The camera being focused on an object at a distance of 6 ft from the first principal plane, find the total depth of focus for the possible movement of the focusing screen, assuming that the tolerance for the disc of confusion in the image is 0.01 in. Hence calculate, from the law of axial magnification, the corresponding range of focus in the object distance.

2. Given the two glasses—

<table>
<thead>
<tr>
<th>Glass</th>
<th>n</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boro-silicate crown</td>
<td>1.5126</td>
<td>62.7</td>
</tr>
<tr>
<td>Dense flint</td>
<td>1.6089</td>
<td>37.3</td>
</tr>
</tbody>
</table>

calculate the focal lengths of the components for an achromatic doublet of power 2 D, and the radii of curvature of the lenses, assuming that the last (flint glass) surface is to be flat, and that the lenses are to be cemented together.

3. Many optical glasses exhibit a relation between their partial dispersions, such as \( n_\lambda - n_d \), and the mean dispersion \( n_p - n_c \), of the type—

\[
n_\lambda - n_d = A_\lambda (n_p - n_c) + D_\lambda
\]

where \( A_\lambda \) and \( D_\lambda \) are always nearly the same for such glasses. Show that thin achromatic doublet lenses made from any such glasses would have a secondary spectrum bearing a fixed ratio to the focal length.

4. Investigate the positions of the Gaussian points of a telescope focused for an infinitely distant object by a normal, unaccommodated eye. What change occurs if the focus is now adjusted so as to project the image into the near point of the eye, assuming this near point to be 25 cm from the eyepiece? (The telescope can be assumed to be constructed from two thin lenses: 2 D for the objective, and 10 D for the eyepiece.)

5. Two thin lenses of respective powers \( F_1 \) and \( F_2 \) for "d" light, and constringencies (or V-numbers) \( V_1 \) and \( V_2 \) respectively, are separated by a distance \( a \) measured along their common axis. Show that the condition that the second principal focus of the system may be achromatic for the C and F wavelengths is:

\[
\frac{F_1}{V_1} = (1 - aF_1)^2 \frac{F_2}{V_2}.
\]
6. It is required to obtain a system, achromatic for the focal length, of two crown glass lenses; the focal length of the system is to be 30 cm. Given that the focal length of one of the lenses is 20 cm, what is the necessary focal length of the other? How far apart should they be mounted?

7. A Ramsden eyepiece must have its components nearer than elementary theory indicates. Assuming a separation of \( \frac{1}{2} \) of the theoretical value, and that each component has a focal length of unity, find the position of the focal and principal planes of the system.

8. Prove that, when two lenses are separated by the algebraic sum of their focal lengths, the linear magnification is independent of the position of the object, and is equal to the reciprocal of the angular magnification for the case of an object at infinity.

9. The eyepiece of a terrestrial telescope of an early type was found to consist of three positive lenses, each of the same focal length \( f' \), separated by equal distances of length \( 2f' \). Discuss the arrangement, and find (preferably by a graphical method) its focal lengths and the positions of its principal planes. Indicate the plane in which the image formed by the objective should be situated, and also possible positions for the cross wires.

10. A negative lens is placed between the objective and eyepiece, and the image is re-focused. Show that the magnification is a linear function of the distance between the negative lens and the eyepiece.

11. A pair of plane mirrors is mounted so that the angle between them is exactly 90º. Discuss the positions of the images (of a single object point) formed by successive reflections in the two surfaces. The angle is now increased by a small amount \( \theta \). Find the result on the image positions.

12. Two coaxial thick systems are separated so that the distance between the second principal focus of the first system and the first principal focus of the second system is a length \( g \). Show that (employing the usual notation) the focal lengths of the complete system are given by

\[
f = -\frac{f_1f_2}{g}, \quad \text{and} \quad f' = \frac{f_1'f_2'}{g}
\]

and that the positions of the principal foci of the complete system are given by

\[
F_1F' = \frac{f_1f_1'}{g}, \quad \text{and} \quad F_2'F'' = -\frac{f_2f_2'}{g}
\]

13. Investigate the positions of the nodal points of a compound microscope in relation to the Gaussian points of the objective and eyepiece.
CHAPTER IV

The Relations between Geometrical and Physical Optics

The subject of the physical phenomena of light and its association with wave motion is fully discussed in works on Physical Optics, to which reference should be made. It is only intended here to recall the steps which have led to principles fruitful also in the theory of instruments.

Theories of Descartes, Fermat, and Malus

In seeking for an explanation of the precise law of refraction, Descartes pointed out that, if the radiation is corpuscular, and the particle on entering a denser medium suffers an increase of its velocity component perpendicular to the surface, but retains the same component velocity parallel thereto, then the formal law of refraction would hold. If \( v \) and \( v' \) respectively are the velocities before and after refraction, the retention of the same velocity component parallel to the surface is expressed by

\[
v \sin i = v' \sin i' \quad (4.01)
\]

so that the velocity must be proportional to the apparent refractive index. Although this explanation could not be linked with any then known property of light, it is valid in the case of "electron optics" where an electron passes through a thin parallel-surfaced lamina of separation between one equi-potential enclosure and another. If the potentials in the first and second enclosures respectively are \( V_1 \) and \( V_2 \), and the electron has been accelerated into the first enclosure from a region of zero potential, the velocities in the two enclosures will be proportional to \( \sqrt{V_1} \) and \( \sqrt{V_2} \) respectively, so that the apparent refractive index of the medium is proportional to the square root of the potential. In the case illustrated in Fig. 71, the equi-potential regions may be enclosed by wire mesh-work so that the electron may pass from one to the other through the interstices. However, similar principles will hold when the particle passes through a field of force in space. The analogy is then to a region of varying refractive index, and the theory of the optics of

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non-homogeneous media has therefore to be considered, a matter largely outside the scope of the present book. See, however, Appendix VI, which may be read after the discussion of the theories of Fermat and Malus below.

In contrast to the theory of Descartes, the principle of Fermat recalled an observation made in ancient times with regard to reflection by Hero, who had pointed out that the path of a ray of light reflected by a plane mirror is the shortest possible distance between any pair of points on the ray path, one before and one after reflection. This is self-evident from Fig. 13 (Chapter I), where the two parts

BP and PQ of the reflected ray are together equal to the straight line B'TQ. Fermat showed that also in the case of refraction the path of a ray refracted at a plane surface would represent the path of least time if the ratio of the sines of the angle of incidence and refraction is equal to the ratio of the velocities of light in the respective media. Referring to Fig. 72, let the two fixed points on a ray path before and after refraction be P and Q, and let the refracting surface be AB. Let the paths PA and AQ make some angles i and i' respectively with the normal. Then if the velocities before and after refraction are c and c' the time t taken on the journey is

\[ t = \frac{PA}{c} + \frac{AQ}{c'} \]

If the perpendicular distance of P and Q from the surface (considered as positive quantities) are \( h \) and \( h' \) respectively, the above expression can be written

\[ t = \frac{h}{c \cos i} + \frac{h'}{c' \cos i'} \]
If the point A is varied while P and Q remain fixed, the angle $i$ will vary also, and
\[ \frac{dt}{i'} = \frac{h \sin i}{c \cos^2 i} + \frac{h' \sin i'}{c' \cos^2 i'} \]
However, since P and Q are fixed,
\[ h \tan i + h' \tan i' = \text{constant} \]
so that (differentiating)
\[ h \sec^2 i \, di + h' \sec^2 i' \, di' = 0 \]
thus
\[ dt = h \sec^2 i \, di \left( \frac{\sin i}{c} - \frac{\sin i'}{c'} \right) \]
It is evident that $dt$ will be zero, and the time will in this case be a minimum, if the bracket term on the right is zero, i.e. if
\[ c' \sin i = c \sin i' \]  \hspace{1cm} (4.02)
which contradicts the hypothesis of Descartes in equation (4.01). The contradiction gave rise to much discussion, but the ideas of Fermat were in accord with the theories of Huygens, and were of course confirmed later by actual measurements of velocities.

Both in the cases of reflection and refraction an actual ray path would be the same independently of the local shape of the refracting surface at the (fixed) point of incidence, provided that the normal is unchanged in direction; in other words the plane surface AB (Fig. 72) may be replaced by any surface tangent to AB at the point A and having the same normal. A similar condition holds in the case of reflection. Referring to the case of reflection at the ellipsoid, it was pointed out on p. 13 that, if the point of reflection B (Figs. 9 and 73) moves around the ellipsoidal surface, the total path and therefore the total time of transit from H to I remains stationary in magnitude. On the other hand, consider reflection at a curved surface touching the ellipsoid at B, but possessing greater curvature. The route HBI would still be a possible one for a ray, but a path (HB₁ + B₁I) via any other point B₁ on the reflecting surface would be shorter. In this case therefore the possible path is the one of maximum length.
Similarly by considering another curved surface touching the ellipsoid at B, but possessing lesser or opposite curvature, it will be found that the possible path for reflection from such a surface will be such as to make the length a minimum. The original idea of Fermat has therefore to be extended by the statement that the possible path is one of minimum, maximum, or stationary time.

Precisely similar conclusions can be drawn in the case of refraction, and the principle can be enunciated in a neat mathematical form. From the equation (4.02) above, it is known that \( c/c' = n'/n \), so that, as \( n = 1 \) for a vacuum,

\[
\frac{1}{c'} = \frac{n'}{c}
\]

The time \( dt \) taken for a disturbance to travel a distance \( ds \) in a medium of refractive index \( n' \) is therefore

\[
\frac{ds}{c'} = \frac{n'ds}{c}
\]

Thus if the path consists of elements \( ds_1, ds_2, \text{ etc.} \), with refractive indices \( n_1, n_2, \text{ etc.} \), respectively, the total time taken is

\[
(n_1ds_1 + n_2ds_2 + \text{ etc.})/c = \Sigma(nds)/c
\]

The value of \( \Sigma(nds) \) is called the "optical path."

The "variation symbol" \( \delta \) now to be used, represents the difference between the actually possible optical path, and another optical path, reckoned along a route geometrically close to it, and separated from it at any point by a distance not exceeding a very small magnitude \( \varepsilon \), and such that the angle between the two routes taken at any point does not exceed the same order of magnitude. Then Fermat's principle states that the variation does not exceed an amount of order \( \varepsilon^2 \). Symbolically,

\[
\delta \{\Sigma nds\} = O(\varepsilon^2)
\]  
\[
(4.03)
\]

In the case of paths in a non-homogeneous medium with continuous variation of refractive index, the equation can be written (provided that \( \varepsilon^2 \) is entirely negligible)

\[
\delta \{\Sigma nds\} \rightarrow 0
\]  
\[
(4.04)
\]

The above formula has, of course, a close relation to the ordinary differential criterion for maximum, minimum, or stationary values of a function. Fig. 74 may be helpful in suggesting the kind of alternative routes envisaged above in the case of continuously variable refractive index, though it obviously exaggerates the variation admissible in an optical case.
Reference must be made to text-books of Physical Optics for the full discussion of Fermat's principle and the consequences which flow from it, and also for another important theorem which can be deduced from it and which was first given by Malus. A statement of it suitable for our purpose says that if a bundle of light rays is characterized by being orthotomic* to some continuous surface in space, then it will still be possible, after any number of reflections and refractions (at continuous surfaces) suffered in common by all the rays, to find another single surface to which all the final ray-paths are orthotomic. As is shown in books on Physical Optics, these surfaces perpendicular to all the rays (orthotomic surfaces) are identified as "wave fronts." Moreover, calculated from one such surface to another along any actual ray track, the optical path has a constant value. See Appendix VI for a brief mathematical discussion.

Since we may (for the present purposes) regard monochromatic light as a periodic disturbance of fixed frequency, the phase of the displacement at any point depends directly upon the time taken in the journey from the origin of the wave motion.

The Theorem of Malus has a consequence important in optical design. Let B (Fig. 75) be a point centre of disturbances in a homogeneous medium; rays diverge from it in straight lines and the orthotomic wave surface W is a sphere. Suppose that the system is so designed that all the rays after transmission intersect in the conjugate point B', then the only surface orthotomic to such a "homocentric" bundle of rays must be a sphere W' centred in B' and situated in the final image space. Now the journey from B to

* Each ray is perpendicular to the plane touching the surface at the point of intersection of that ray.
W by all routes is equal, and likewise the journey from W' to B'; hence with the addition of the theorem of Malus the total time taken along every possible route from B to B' is equal, and all disturbances must arrive in the same phase. Thus merely by designing the system so as to bring all the rays to one focus, we automatically control the optical paths. Equally, by so designing a system that the optical paths from object point to image are all identical we shall ensure that all the rays will pass through the image point. These two approaches to optical design are thus complementary.

For example, suppose a plane wave AB from an infinitely distant object point on the X-axis is incident on a curved refracting surface touching the YZ-plane (Fig. 76) at the origin. If the X-axis is the axis of symmetry we need only consider the X,Y-section of the surface, in which the coordinates of P on the ray BP parallel to the axis are \(x, y\).

Then if \(F'\) is an axial point at a distance \(l'\) from A, the condition that all optical paths are equal from the plane wave to \(F'\) is (writing \(n\) and \(n'\) for the refractive indices before and after refraction)

\[
n' \cdot l' = nx + n' \{y^2 + (l' - x)^2\}^\frac{1}{2}
\]

Rearranging, the equation takes the form

\[
x^2 \left(1 - \left(\frac{n}{n'}\right)^2\right) - 2 \left(1 - \frac{n}{n'}\right) l'x + y^2 = 0
\]

On comparing this with the usual equation of an ellipse referred to the vertex as origin, i.e.

\[
x^2 \left(\frac{b^2}{a^2}\right) - 2 \left(\frac{b^2}{a}\right) x + y^2 = 0
\]

we see that the equation is of the same type, and that the eccentricity, corresponding in the latter case to \(1 - (b^2/a^2)^\frac{1}{2}\), is \(n/n'\). The curve of section is an ellipse, provided that \(n' > n\), and the major axis is \(2l' \cdot n'/(n + n')\). This result agrees with that proved on p. 14, but it has been obtained by direct analysis, a matter which would have been much more difficult with the former approach.

Unfortunately the difficulty of supplying direct and useful solutions of such simple kinds rapidly increases when more conditions have to be satisfied.
**Extension of Fermat’s Theorem**

Let PQ (Fig. 77) be a ray path in an isotropic though not necessarily homogeneous medium, and let W be the locus of an associated wave front at the moment of its intersection with Q; the local path of the ray at Q is perpendicular to W. Let $Q_1$ be a point on W such that the interval $QQ_1$ is small of the first order; then it can be shown that an optical path PQQ\(_1\) associated throughout with PQ in the way defined above for Fermat’s theorem will differ from the optical path PQ by a quantity not exceeding the second order of smallness. This is proved in textbooks of Physical Optics.

This can be expressed in the same symbolic way

$$\delta\int nds \to 0$$

where the $\delta$ indicates the variation just mentioned. An example of the usefulness of this theorem will now be given.

**The Physical Basis of the Smith–Helmholtz Invariant**

The extension of Fermat’s theorem sometimes gives a clearer grasp of the physical basis of a proposition in Geometrical Optics.

Let AL, AM (Fig. 78) be two rays crossing at A and including between them a very small angle $d\alpha$; let them enter an optical system S, of which the axis of symmetry lies in the plane of the diagram. Let $L'A'$, $M'A'$ be their paths on emergence intersecting in a point $A'$. Related to the two rays AL, AM we might have wave fronts shown by short traces as AB, AD, to which the rays at A are respectively
normal. Thus (in the condition mentioned) there will be corresponding wave fronts of which the tangents at A' will be perpendicular to L'A' and M'A' respectively; let A'B' and A'D' be short traces of these wave fronts; they include between them at A' a very small angle dξ'. Neither A nor A' are to be taken near a focus of the wave fronts.

Let AB be small of the 1st order; at B draw the normal BDH cutting the second wave front in D, and trace this ray through the system; it will cut A'B' in B', say, and the second wave surface in D'. If A'B' were found not to be small of the 1st order, it would in general be possible to reduce AB until it were so. We will suppose this to be done. In any case (by Malus's theorem) the complete optical path from A to A', denoted by AL → L'A', is equal to BH → H'B'.

Trace a ray DJ through D normal to AD, and onwards through the system; it will finally cut the second wave surface in a point D'' so close to D' that D'D'' is negligible in comparison with A'D'; if this were not so, we could in general reduce the angle dξ until the condition is obtained. Then by the extension of Fermat's theorem, the optical path DJ → J'D'' is equal to the optical path DH → H'B'D', which is in turn equal to AM → M'A'. This by Fermat's theorem is equal to AL → L'A' and thus as above to BH → H'B'.

Then since \((BDH \rightarrow H'B') = (DH \rightarrow H'B'D')\)

optical path BD = optical path B'D'

Since AB and A'D' are 1st-order quantities we may neglect any curvature in the wave fronts. Writing \(n\) and \(n'\) as the initial and final refractive indices, we thus have (treating the numerical magnitudes as positive)

\[ n(AB) \, dξ = n'(A'B') \, dξ' \]

where AB and A'B' are corresponding distances on a wave front between the intersecting points of rays; and by the supposition above the crossing points A and A' of the rays must be conjugate at least to the extent that some rays intersecting in A meet again in A'. Compare equation (1.41).

There is nothing in the above argument to show that there must be true conjugate points B and B' near to A and A' respectively; on the other hand, if there are such points, then the theorem tells us what the magnification A'B'/AB must be. It was shown in the paraxial theory that true conjugate points B and B' could be expected to be found in the planes perpendicular to the axis passing
through axial conjugate points A and A' respectively, and the Smith–Helmholtz relation (1.41)—

\[ nh\beta = n'h'\beta' \]

therefore applies to such a case; but it has other applications also.

**Angular Magnification due to a Prism**

Consider a group of parallel rays traversing a prism in its principal section. The perpendicular distance between two rays (which can be considered to be measured in the wave front) will in general be altered after transmission from \( b \) to \( b' \) say. If the initial and final values of the refractive index are \( n \) and \( n' \), the foregoing theorem enables us immediately to write down the rate of change of direction of the emergent ray consequent on a small change in the angle of incidence, since by the arguments of the foregoing paragraphs

\[ nb \, d\alpha = n'b'd\alpha' \]

Note that in the case of refraction by a prism \( d\alpha \) and \( d\alpha' \) will have the same sign if the rotation of the ray has the same sense. Thus if the angles of incidence and refraction are reckoned positive as shown in Fig. 68 we must then write—

\[ nb \, di_1 = - n'b'di_2 \]

If \( n \) and \( n' \) are equal and \( b \) and \( b' \) are also equal in the symmetrical position, it is easily seen that the rate of change of deviation is zero in this case, and further that the deviation will be a minimum rather than a maximum or stationary; for since the breadth of the emergent beam will be zero if the rays have grazing emergence, it is also easily seen that near this position there will be a relatively large numerical ratio of \( di_2/di_1 \); conversely, the ratio is very small if the incident rays are almost parallel to the surface, provided always that the angle of the prism is small enough to allow transmission by the further surface with incidence at an angle less than the “critical” value.

The relationship discussed in the foregoing paragraphs is seen to be quite independent of the nature of the instrument by which the change in the diameter of the beam is made; it may for example be a telescope—and the relation between the angular magnification and the ratio of the diameter of the pupils of the system follows immediately.

**Dispersion of a Prism**

A plane wave shown in section by AE (Fig. 79) with associated rays AB and EF, is refracted by a prism. The plane wave front DH
emerging again into air is normal to the refracted rays CD and GH. If the wavelength of the light is changed slightly, the refractive index of the prism suffers a corresponding change from \( n \) (as compared with unity for air) to \( n + \delta n \). The refractive index of the air remains unchanged. The exact paths of the rays are identical along AB and EF (for the wave front section AE is supposed unchanged); but they vary after refraction at the first surface, and the final wave front section will not be parallel to DH. Suppose it is represented by some other line DI through D, suggested by the broken line in the figure, cutting GH in I. Now, although we do not know the new exact ray paths, the original ray tracks taken as far as the new wave

\[
\text{Fig. 79.}
\]

front can be regarded as associated paths (defined as in Fermat's theorem and its extension) by means of which we are allowed to calculate the optical paths from the first wave front to the new one (with negligible error) by finding \( \Sigma n ds \) along them. Let us write \( t_1 = BC \), and \( t_2 = FG \) in the figure.

Now for the original wavelength (theorem of Malus).

\[
AB + nt_1 + CD = EF + nt_2 + GH
\]

and also, for the new wavelength,

\[
AB + (n + \delta n)t_1 + CD = EF + (n + \delta n)t_2 + GI
\]

Subtracting

\[
(\delta n)t_1 = (\delta n)t_2 - IH
\]

or

\[
IH = A\delta\alpha = \delta n(t_2 - t_1)
\]

where \( \delta\alpha \) is the change of angle associated with the change of refractive index, and \( A \) is the breadth of the transmitted beam associated with the difference \( (t_2 - t_1) \) of the path-lengths in the prism. Hence, in the limit, for small changes of refractive index,

\[
\frac{d\alpha}{dn} = \frac{t_2 - t_1}{A} \quad \quad \text{... (4.05)}
\]
This is a typical example of the way in which a result belonging to the field of geometrical optics can be obtained from considerations of optical path.

**Physical Images of Points and Slits.**

In the purely geometrical theory, the image of a point source may be a point also, but in practice, owing to the finite wavelength of light, the image of an object of negligible dimensions (or of negligible angular subtense at the image-forming system) always has finite dimensions in space. Thus if the real image of such a "point source" is formed by a lens (the ray-reunion in the image point being perfect)

![Figure 80](image)

and if the aperture of the converging beam (Fig. 80) is restricted by a symmetrical rectangular diaphragm XY, the relative distributions of amplitude and intensity in a line B'C in the geometrical image plane and parallel to one of the edges of the diaphragm are represented by—

\[
\text{amplitude} = \text{constant} \times \left( \frac{\sin z}{z} \right)
\]

\[
\text{and intensity} = \text{constant} \times \left( \frac{\sin z}{z} \right)^2
\]

where \( z = \frac{2\pi}{\lambda} \rho \sin \alpha \)

\[ (4.06) \]

\[ (4.07) \]

and \( \alpha \) is the angular semi-aperture of the beam (supposed small), \( \rho \) is the distance measured in the above line B'C from the centre of the image, and \( \lambda \) is the wavelength. These functions are plotted against \( z \) in Fig. 81, from which it is seen that the successive maxima rapidly diminish. The *phase* may be said to suffer abrupt change of \( \pi \) radians at the regions of zero displacement, but is otherwise constant. The
conclusions are modified if the aperture is large or the amplitude in the convergent wave front is not uniform.

If the aperture is circular, the image also has circular symmetry, and the amplitude and intensity are then given by

\[
\text{amplitude} = \text{constant} \times \frac{2J_1(z)}{z}
\]

\[
\text{intensity} = \text{constant} \times \left( \frac{2J_1(z)}{z} \right)^2 .
\]

(4.08)

![Graph showing amplitude and intensity calculations.](image)

**Fig. 81.** Case of rectangular aperture.

Broken curve \( C = (\sin z)/z \)

Full curve \( C^2 \)

where* \( J_1(z) \) is the well-known Bessel function (somewhat similar to a cosine curve with diminishing amplitude). Tables of numerical values of the function are given in mathematical books of reference, so that it is a convenient means of expression. The curves are illustrated in Fig. 82. Alternation of phase takes place in the same way as described above. The calculation of the distribution of light in the diffraction figure was first performed by Sir George Airy in 1834, and the phenomenon is usually called the "Airy disc."

* The limit of \( J_1(z)/z \) when \( z = 0 \) is 0·5, so that the normalized expression giving unity at the centre is usually written \( (2J_1(z))/z \).
Fig. 82. Case of circular aperture.
Broken curve \( C = \frac{2J_1(z)}{z} \)
Full curve \( C^2 \)

Fig. 83. Condition for "limit of resolution."
Broken curve—sum of intensities
Full curves—separate intensities
Resolving Power of Spectroscope

If the image of a slit of negligible width is formed by a lens system with an aperture limited by a rectangular diaphragm (often used in spectroscopes) the slit being parallel to a diaphragm edge, the relative distribution of amplitude, perpendicular to each line image, in the focal plane is represented, as mentioned above, by the sine function \((\sin z)/z\). If two adjacent spectrum lines of equal intensity are so situated that the central maximum intensity of the first falls on the first zero intensity of the second, the sum of the intensities suffers a slight drop between them, as suggested in Fig. 83. If the lines are only a little nearer together, this drop will not be found and the lines will not be “resolved” in the previous sense. The conventional “resolving limit” is taken to be the condition just mentioned. In this case the linear separation of the lines in the focal plane is such as to make \(z = \pi\), i.e.

\[
\rho = \frac{\lambda/2}{\sin \alpha} \quad \cdots \quad (4.09)
\]

If the (small) aperture of the lens system limiting the cone of rays has a diameter \(A\) and the image distance measured therefrom is \(f'\), the above formula becomes

\[
\rho = \frac{\lambda/2}{A/2f'} = \frac{f'\lambda}{A}
\]

and thus \(\delta\alpha\), the angle between the pair of spectrum lines just resolved is

\[
\delta\alpha = \rho = \frac{\lambda}{f'} = \frac{\lambda}{A} \quad \cdots \quad (4.10)
\]

The “resolving power” of the spectroscope is often written \(\lambda/\delta\lambda\), where \(\delta\lambda\) is the wavelength difference corresponding to the above limit of resolution. Hence, from the last equation,

\[
\frac{\lambda}{\delta\lambda} = A \frac{\delta\alpha}{\delta\lambda}
\]

but from (4.05) above,

\[
\frac{\delta\alpha}{\delta\lambda} = \frac{\delta\alpha}{\delta n} \frac{\delta n}{\delta\lambda} \simeq \frac{t_2 - t_1}{A} \frac{\delta n}{\delta\lambda} \quad \cdots \quad (4.12)
\]

therefore

\[
\frac{\lambda}{\delta\lambda} \simeq (t_2 - t_1) \frac{\delta n}{\delta\lambda} \quad \cdots \quad (4.13)
\]

It is seen that the limit of resolving power depends solely on the effective difference of path within the prism for the rays at the
extremes of the aperture, and on the dispersive properties of the
glass, but the above calculation is only valid when the slit is so fine
that the width of its geometrical image is small in comparison with
the distance between the flanking regions of zero intensity. It is
often necessary, however, where the brightness of the spectrum
must be high, to use slits of such a width that the effective distribu-
tion of intensity in the image is greatly modified and the resolving
power found in practice may be correspondingly reduced, especially
if photography is used to record the image (see p. 144).

**Physical Resolving Limits of Telescopes**

The simplest case for consideration is that of a "double star," each
member having equal magnitude and negligible angular subtense
in the field. It is often possible to correct the objectives of telescopes
(with circular apertures) so completely that the distribution of in-
tensity in each of the single star images is well represented by the
Airy disc (see equation (4.08)). Reference to a table of Bessel func-
tions shows that $J_1(z)$ has its first zero value when $z \simeq 3.83$. If the
centres of the images of two stars are separated by this distance in
the focal plane of the objective, the sum of the intensities along the
line joining them shows (as described above) a slight dip which
would soon disappear if they were moved appreciably closer to-
gether. This separation, when the centre of one image falls on the
first dark minimum of the next, is taken conventionally as the
resolving limit of the objective, though indeed the conditions
under which the image would be *visually recognized* as double need
further discussion (see p. 134). From the definition of $z$ in (4.07),
we have

$$\frac{2\pi}{\lambda} \rho \sin \alpha \simeq 3.83$$

or

$$\frac{\rho}{\lambda} \sin \alpha \simeq 0.61 \ . \quad \ldots \quad \ldots \quad (4.14)$$

When $\alpha$ is small, as is usual in practice, it may be replaced by $A/2f'$,
where $A$ is the diameter of the diaphragm of the objective (supposed
of negligible thickness) and $f'$ is the back focal distance of the image
plane. As the nodal point of the image space may (under the above
assumption) be taken to be coincident with the objective, the
angular separation $\omega_0$ of the stars in the field of view is $\rho/f'$; thus

$$\text{angular resolving limit} = \omega_0 \simeq \frac{1.22\lambda}{A} \ . \quad (4.15)$$

5—(T.785)
Provided, therefore, that a well-corrected telescope has an unobstructed circular aperture the angular resolving limit is simply dependent on the wavelength of light, and on the diameter of the aperture. If the *resolving power* is defined as the reciprocal of the angular resolving limit, it will be seen to be proportional to the number of wavelengths of light in a distance equal to the diameter of the objective.

**Physical Resolving Limit of the Microscope**

The object plane will, in the first instance, be assumed to contain two point sources of light close to the axis. They are supposed to radiate non-coherent light uniformly into the aperture of the objective. Each emergent beam travelling toward the internal image plane is limited by the circular diaphragm of semi-diameter \( y \) situated at a distance \( l' \) (say) in front of the image. Provided that the objective is well corrected, the image of each source then approximates to an Airy disc distribution of the usual type, in which the radius of the first dark ring is represented by

\[
\rho \approx \frac{0.61\lambda}{y} \approx \frac{0.61\lambda}{\sin U'}
\]

where \( U' \) is the (usually small) angle of convergence of the extreme ray with respect to the principal ray; then if the two sources are so close that their images are situated (as in the case, discussed above, of the telescope) at a distance \( \rho \), the conventional limit of resolution is attained. Writing \( m \) for the transverse linear magnification of the objective, the resolving limit may, therefore, be written \((1/m) (0.61\lambda/\sin U')\) but this formula can be put into a more useful form by the use of the "optical sine theorem," which must now be explained.

The paraxial form of the Smith–Helmholtz invariant was

\[
nku = n'k'u'
\]

or

\[
nu = n'u'(m)
\]

where \( m \) is the linear magnification. Now in most practical compound microscopes it will be found that \( u' \) is of the order of 0.04 radians. Thus even with a first magnification of only ten, the value of \( nu \) (supposing \( n' = 1 \)) would perhaps be 0.4. If \( n = 1 \), this means that \( u \) would rise to 23°, which is far outside the paraxial range. Hence the above paraxial form of the equation is invalid, and a relation must be sought which remains true when \( u \) and/or \( u' \) become large.
The Optical Sine Theorem

This affords another example of a way in which Fermat's theorem can be useful in finding a general relation independently of details of ray paths.

Consider an optical system (Fig. 84) with its axis of symmetry BAA'B', where A and A' are the poles of the first and last surfaces. The initial and final refractive indices are \( n \) and \( n' \) respectively. A point \( P \) is situated in a plane touching the first surface at \( A \); then if \( B \) is an axial object point, rays from it are supposed to pass into the system through various points such as \( P \). Let the ray \( BP \) be traced through the system and let it intersect, at \( P' \), a plane touching the last surface at \( A' \). From the symmetry of the system, \( A'P' \) will be parallel to \( AP \). This ray from \( B \) finally intersects the axis in the point \( B' \). Let the value of the optical path between \( P \) and \( P' \) for this ray be written as \( \Omega \), say, and let the system be so free from aberration that the optical path \( BPP'B' \), whatever the position of \( P \) or its radial distance from \( A \) (subject to the transmission of the ray) remains constant. Then \( B' \) is the physical image of \( B \).

Take a point \( C \) at a very small perpendicular distance \( BC = \delta \) from \( B \). We shall seek the condition that a sharp image of \( C \) may be found at some point \( C' \) in the image plane containing \( B' \). Again from symmetry, \( C' \) will be sought in the axial plane containing \( C \). Let \( B'C' \) (to be found if possible) be written \( \delta' \).

Now consider a new ray \( CP \) which is derived from \( C \) and enters the system at \( P \); its further path cannot yet be exactly specified (though it will intersect \( C' \) if the latter is the physical image); but we know that by making \( \delta \) sufficiently small, the point of intersection of the new ray in the second tangent plane may be made to
approach $P'$ as closely as we please. Thus by Fermat's theorem the optical path $O_p$ from $C$ to $C'$ can be calculated by adding together three parts as follows—

$$O_p = n(CP) + \Omega + n'(P'C') \quad . \quad (4.16)$$

Let the Cartesian coordinate of $P$ in the first tangent plane (the $Y$-axis being parallel to $BC$) be $y, z$, then if $BA = l$,

$$CP^2 = (y - h)^2 + z^2 + l^2$$

while

$$BP^2 = y^2 + z^2 + l^2$$

Hence

$$BP^2 - CP^2 = 2yh - h^2$$

However, if $h$ is a 1st-order quantity, $h^2$ may be neglected in comparison with the other term; thus

$$BP - CP \simeq \frac{2yh}{BP + CP} = \frac{2yh}{2BP - (BP - CP)}$$

$$= \frac{yh}{BP} \left(1 - \frac{BP - CP}{2BP}\right)^{-1}$$

By substituting for $BP - CP$ in the bracket term in the last expression and expanding the bracket, it is clear that in neglecting terms in $h^2$ and higher powers we may write*

$$BP - CP = \frac{yh}{BP}$$

If the distance $AP = \rho$, and $\widehat{PAY} = \phi$, we may write $y = \rho \cos \phi$ so that

$$CP = BP - h \left(\frac{\rho}{BP}\right) \cos \phi$$

$$= BP + h (\sin U) \cos \phi \quad . \quad (4.17)$$

where $U$ is the angle $\widehat{ABP}$, considered as having a numerically negative sign in accordance with the sign convention.

In a similar way, we may show that

$$P'C' = P'B' - h' (\sin U') \cos \phi \quad . \quad (4.18)$$

where $U' = \widehat{P'B'A'}$ (considered positive) and $h'$ is correspondingly negative as suggested in the diagram. Substituting (4.17) and (4.18) in the expression (4.16) for $O_p$, we obtain

$$O_p = n(BP) + nh (\sin U) \cos \phi$$

$$+ \Omega + n'(P'B') - n'h'(\sin U') \cos \phi$$

* This result is easily seen to be approximately true when $BP$ and $CP$ are nearly equal, but it is often important to make sure what approximations are actually involved.
GEOMETRICAL AND PHYSICAL OPTICS

As P travels round A in a circular locus, BP and B'P' remain constant and \( \Omega \) remains constant from symmetry. Hence the total coefficient of the variable (cos \( \phi \)) must vanish if \( O_p \) is to be constant, i.e.

\[
nh \sin U - n'h' \sin U' = 0
\]

Written in the symmetrical form,

\[
nh \sin U = n'h' \sin U' \quad \quad \quad (4.19)
\]

this is called the optical sine relation. It expresses the magnification in the image formation* effected by a zone, in the system, defined by a constant value of \( U \). However, if all zones of the system are to give the same magnification (so that the definition of image points at short distances from the axis may be good), the condition is

\[
\frac{n \sin U}{n' \sin U'} = \frac{h'}{h} = \text{constant} \quad \quad \quad (4.20)
\]

This relation is called the “optical sine condition.” It has to be fulfilled by microscope objectives within close limits.

The Resolving Limit of the Microscope (continued)

We return now to the expression for the resolving limit of the microscope. It was shown above that the separation of points in the image plane, corresponding to the resolving limit, was \( 0.61 \lambda / \sin U' \). From (4.19) the corresponding separation \( h \) of conjugate points in the object plane is (if \( n' = 1 \), as usual)

\[
h = \text{resolving limit} \approx \frac{0.61 \lambda}{n \sin U} \quad \quad \quad (4.21)
\]

where \( U \) is the maximum possible initial angle of divergence between the axis and a ray which, starting from an axial object point, can reach the image. The product \( n \sin U \) is known as the “numerical aperture” of the objective. This, then, is the expression for the resolving limit for two point sources, the light being supposed to be “incoherent.”

As explained in works on physical optics, “coherence” implies a steady phase-relation between the sources. Self-luminous object points would be “incoherent” in this sense. However, the compound microscope system is usually supplemented by a condenser or condensing-lens system, by means of which objects on the stage can be illuminated by light derived from a suitable lamp in which the source of light has an appreciable size. It is beyond the scope of geometrical optics to discuss the optimum arrangement of the illumination for various types of object, but it will be understood

* But only for images so small that \( h^2 \) is negligible.
that if the source were extremely small and the objects on the stage were represented by two very small apertures illuminated by parallel rays from the condenser, the light would approach complete coherence and the condition for resolution would be markedly different. The image is, in such a case, more difficult to interpret. However, the degree of coherence can be diminished by increasing the angular aperture of the bundle of illuminating rays passing through any part of the object plane, either by increasing the aperture of the condenser or increasing the size of the source. It can be shown that if the numerical aperture of the condenser is equal to that of the objective the conditions at the conventional resolving limit are very largely equivalent to complete incoherence; thus the formula above is valid for such a case.

The great pioneer of the theory of illumination in the microscope was Abbe of Jena (1840 to 1905). Owing to its technical importance there is now a vast literature on the subject.*

**Improvement of the Microscope**

From the expression (4.21) it appears that the resolving limit will grow smaller and smaller as the numerical aperture increases, or as the wavelength of the light diminishes.

The use of *homogeneous immersion*, by which the object is imbedded in a medium of comparatively high refractive index, allows of the increase of numerical aperture by increase of \( n \). Again, the improvement of the correction of the objective, so that it will collect and focus rays diverging at greater and greater angles from the axis, means a corresponding increase of \( \sin U \). Lastly, the use of shorter and shorter wavelengths (using blue or ultra-violet light) is possible, provided that the image is photographically or otherwise recorded. The limit to progress in this last direction is caused by the opacity of the various optical media for the far ultra-violet radiations.

The possibility of a great increase of numerical aperture, even when using spherical surfaces, by increasing both \( n \) and \( U \) is offered by the principle of “aplanatic” refraction. The sphere (Fig. 85) of centre \( O \) and radius \( r \) has refractive index \( n \) and is embedded in a medium of refractive index \( n' \). The points \( B \) and \( B' \) are situated on a common radius so that

\[
OB = r(n'/n) \\
OB' = r(n/n')
\]

* References to books and papers may be found in the present writer's *Technical Optics*, Vol. II (London: Pitman). The question of the depth of focus of a microscope objective is briefly discussed in Appendix II, p. 198, below.
Let a ray BP cut the sphere in P (where the normal is OPN); let \( \widehat{OBP} = U \); join B'P and produce it to some external point Q; let QB'O = \( U' \).

The two triangles with apices at O, i.e. BOP and POB', are now similar, since they have a common angle, and

\[
\frac{OP}{BO} = \frac{B'O}{OP} = \frac{n}{n'}
\]

![Diagram](image)

Fig. 85.

Then \( \widehat{BPO} = U' \) and \( \widehat{B'PO} = U \). Looking now at the triangle BOP,

\[
\frac{n}{n'} = \frac{OP}{BO} = \frac{\sin U}{\sin U'} = \frac{\sin B'PO}{\sin B'PO}
\]

and thus the angles \( \widehat{BPO} \) and \( \widehat{NPQ} (= B'PO) \) agree with those prescribed by the law of refraction independently of the position of P. This means that all rays, from B, refracted at the further surface of the sphere, have paths diverging from one virtual image B'.

Since

\[
\frac{\sin U}{\sin U'} = \frac{n}{n'} = \text{constant}
\]

the optical sine condition is fulfilled, and sharp image formation can be expected for object points slightly off the axis in the plane of the object.

Fig. 86 shows how the principle can be used in connexion with the front lens of an immersion microscope objective. The object can be embedded in a medium (e.g. Canada balsam) which has practically the same refractive index as the lens. The actual front lens of the objective has its front surface plane, and the medium between this surface and the cover glass may be an oil (e.g. cedar-wood oil) of which the refractive index is again practically equal to that of the
glass on each side. The refracting surfaces can thus be brought into such a position that the object point is at one of the internal aplanatic points; the divergence of the cone of rays then suffers a great reduction on refraction. The rays can then be made to enter a second lens (Fig. 87) of which the first surface is so placed that they now enter it "normally"; once in the glass they can be given a second refraction of the aplanatic type by so arranging that the outer aplanatic point for the first refraction is (geometrically) the inner aplanatic point for the second. Thus by two aplanatic refractions the ray divergence is made so modest that the real image may now be formed by cemented lenses of more ordinary types.
Considerable "under-corrected" chromatic aberration arises at these two aplanatic refractions, and this has to be compensated by the chromatic "over-correction" given to the later components.

Microscope Condensers

In order to secure the optimum resolution of the microscope it is sometimes required that the numerical aperture of the illuminating rays passing through a point of the object shall approach or equal that of the objective. This can obviously be secured by using a reversed objective for the purpose; however, in most cases the optical correction need not be nearly so perfect owing to the appreciable size of the usual sources of light; and a comparatively simple system with fewer surfaces (though often similar principles) will suffice. Such a lens system is suggested in Fig. 88; it would have large residual aberrations.

EXERCISES IV

1. An electron has been accelerated from zero potential, and is moving in an equipotential enclosure, the potential being $10^4$ volts. It now enters a series of four contiguous equipotential regions, the boundaries of which are plane and parallel, the angle of incidence on the first boundary being 60°. The potential increases by 1,000 volts at each boundary. Find the angle of refraction with which the electron enters the fourth region, assuming that its passage is completely unobstructed.

2. A transparent medium has a refractive index which is symmetrical about a central point. Show that if $p$ is the length of the perpendicular from the centre to the tangent to a ray path (the tangent being drawn at the point where the refractive index is $n$), then

$$np = \text{constant}$$

3. A surface of revolution separates a medium of refractive index 1·0 from another of refractive index 2·0. An axial object point in the first medium, distant by 4·0 units from the pole of the surface, has an image in the second medium at a distance of 1·0 unit. There is no aberration. Find the equation of the section of the appropriate refracting surface.

4. Calculate the angular dispersion in radians per millimicron in the neighbourhood of $\lambda = 500 \mu\mu$, of a flint glass prism for which the constants in the simple Hartmann dispersion formula (Appendix IV) are $n_0 = 1·60$, $c = 12·00$, and $\lambda_0 = 200$, given that the prism is equilateral with sides of length 2 in., of which the full aperture is used at minimum deviation.

5. Calculate the numerical resolving power $\lambda/\delta\lambda$ of a spectroscope using the prism specified in Exercise 4, and in the same region of the spectrum. Given
that the instrument is equipped with telescope and collimator lenses, each of focal length 10 in., find the slit width, of which the "geometrical image" would have one-quarter the width of the central "physical image" measured between the first diffraction minima on each side.

6. A thin lens of focal length 10 in., has a circular aperture with a diameter of 0.5 in., and has negligible aberrations for axial object points at any distance not less than 10 in. Find the angular resolving limit (in degrees) at the centre of the lens for point objects near the centre of the field at the following distances from the lens: infinity; and 20 in. The wavelength of light is to be taken as 0.5 × 10⁻⁴ cm.

7. A microscope objective is designed for objects immersed in a liquid of refractive index 4/3. The extreme ray which can enter the lens from an axial object point makes an angle of 60° with the axis. The lens is used with ultraviolet light with wavelength 2,560 Å. Find the number of lines per millimetre in the finest high-contrast structure which could, in principle, be resolved, assuming the illumination to be perfectly incoherent, and symmetrical with the axis.

8. A telescope objective satisfies the sine condition when the object point is at infinity. Show that the locus of the intersection point of the initial and final directions of rays parallel to the axis in the object space is a sphere.

Discuss the relation of this result to the validity of the conception of principal planes in the Gaussian sense.

9. A homocentric bundle of rays, proceeding towards an axial point at a distance of 10 cm beyond the pole of a spherical refracting surface, suffers aplanatic refraction, and is there directed toward a point at 6.5 cm distance. Find the radius of curvature and the relative refractive index.

10. Supply short answers to the following queries—
(a) It is required to measure the diameter of the exit pupil of a compound microscope. How would you supply the light entering the objective if no condenser is available?

(b) An oil immersion lens of N.A. 1.4 is used to view an object mounted in Canada balsam in a normal microscope slide. No immersion oil being available, the space between the flat front surface of the objective and the flat top of the cover slip on the slide is almost filled by a thin plate of glass having the same refractive index as the oil. Would this be satisfactory?
CHAPTER V

The Observation and Recording of Images

The performance of optical instruments has necessarily to be discussed in relation to the means employed for making use of the image obtained.

The Eye

The structure and properties of the eye impose certain requirements on the design of visual instruments. An eye is roughly globular in form except for the transparent protuberance of the cornea in front (Fig. 89). The cornea is a continuation of the outer shell of the eye; the opaque whitish part of the main covering is known as the sclerotic. The optical system owes the major part of its dioptric power to refraction at this highly curved corneal surface. The interior is filled with liquid or jelly-like transparent materials (the aqueous and vitreous humours), but the so-called crystalline lens, of a harder and more highly refracting substance (though not crystalline in the ordinary sense), is held by the muscular system of the
ciliary muscle and ciliary process at a short distance behind the cornea. The aperture of the lens is restricted by the iris immediately in front of its outer surface.

This refracting system projects images of exterior objects on to the retina, the sensitive surface which lines the interior of the globe. The sensitive elements are the rods and cones—spindle-shaped bodies, the smallest of which are only about 2 microns in diameter; they form a close mosaic, their long axis being perpendicular to the retinal surface. They are connected by nerve fibres to the brain through the channel of the optic nerve.

The eye moves in the bony cavity of the skull called the orbit, and is set deep enough behind the frontal bones of the forehead to obtain protection. The eyelids screen the cornea, and distribute a film of the lachrymal fluid (secreted by the tear glands) in order to maintain a clear refracting surface. The direction of the visual axis is controlled by a group of six muscles which can move it into various directions with relation to the orbit; but the head is, of course, usually turned into a fairly convenient direction. The field of sharp vision of an eye is very small, probably because the cones capable of giving it are located only close to a restricted region of the retina called the fovea. Certain substances exist in the retina which are readily decomposed by light; and the decomposition is accompanied by photo-electric effects in the receptors, capable of originating the electrical nerve pulses which result in the sensation of light.

Accommodation

In contrast to the focusing of a camera (which is, of course, usually carried out by adjustment of the distance of the lens from the screen), focusing is effected in the eye by a change in the curvature of the surfaces of the crystalline lens, which is of a more or less elastic nature. The change can thus be produced by alterations in the tension of the ciliary muscles.

Schematic Eye

It is beyond the scope of this book to go into details of the optical construction of the eye. Owing, however, to the preponderance of the power of the refraction at the cornea, its action (when unaccommodated) is somewhat similar to that of a system with one spherical refracting surface, which would have coincident nodal points at the centre of curvature and coincident principal points at the surface. The relevant data are given in connexion with Fig. 90, and are adequate for rough introductory calculations. More representative data are given in Example 8, p. 70, from which optical constants
can be calculated, but it should be understood there is a range of statistical variation.

**Visual Acuity**

The diameter of the retinal elements (cones) near the seat of most distinct vision in the retinal mosaic being about 2 μ, it might be

![Fig. 90. Simplified eye.](image)

\[
\begin{align*}
AP &= 2.2 \text{ mm} & NF' &= 15.0 \text{ mm} \\
PF &= -15.0 \text{ mm} & AF' &= 22.2 \text{ mm} \\
PF' &= 20.0 \text{ mm} & AF &= -7.2 \text{ mm} \\
AN &= 7.2 \text{ mm} & \\
\end{align*}
\]

supposed that a condition that two bright images might be seen as resolved would be that the stimulation of a particular cone falling between two such images would be less than the stimulation of the flanking cones.

Where the elements are packed fairly closely together, they will tend to form a rough hexagonal mosaic structure as suggested in Fig. 91, though the actual evidence of photomicrographs does not confirm the existence of anything approaching a regular honeycomb pattern; thus the centres of the flanking cones (XX) would be about 4 μ apart. This distance on the retina would subtend an angle of 0.004/15 radians or about 56 seconds of arc. This value corresponds in fact fairly well to the usually accepted limit of angular acuity of an eye. Hooke estimated as long ago as 1671 that the limiting resolvable separation for an object such as a double star was about one minute of arc. However, experiments with well-illuminated high-contrast gratings (objects with equi-spaced bars and spaces) show that some individuals can see the structure when a grating
element (centre to centre of adjacent bars) subtends considerably less than one minute, so that the full explanation is not so easily given.

The limiting angle increases \textit{greatly} when the contrast of the object diminishes, or when the illumination is low.

It is remarkable that the limit suggested by the retinal “grain” is similar to that which would be required by the wave nature of light. The diameter of the pupil in the unaccommodated eye is dependent upon the level of illumination of the field; in bright daylight it may shrink to 2 mm or thereabouts, and by using the formula (4.15) it may be shown that the angular resolving limit corresponding to a wavelength of $0.5 \times 10^{-4}$ cm is just over one minute. It does not, however, follow that a better acuity should necessarily be attained if the diameter of the pupil is enlarged, since the aberrations and irregularities of the optical system are increasingly prominent. In spite of a great deal of research many problems are still outstanding. The maintenance of the optimum resolution in spite of the retinal grain contrasts very favourably with the experience met in photography, etc. (see below).

In any case, the recognition of detail subtending an angle of about one minute involves a considerable effort of attention. Hence in any optical system the magnification used should be such that the smallest high-contrast detail is seen (as mentioned above) under an angle of four to six minutes—or even more if the contrast is low, or the illumination poor, or both.

\textbf{Relative Luminosity in the Spectrum}

The refracting system of the eye is not achromatic, and yet the optimum resolving power is apparently close to the theoretical limits. One of the main reasons for this is found in the highly selective character of the response of the eye to the various wavelengths of light. The approximate relative luminosity (see p. 162) of the parts of a spectrum with uniform dispersion ($\frac{ds}{d\lambda}$ is constant, where $s$ is the displacement in the focal plane and $\lambda$ is the wavelength) and uniform energy ($\frac{dE}{ds}$ is constant, where $E$ is the energy) is shown in Fig. 92, from which it appears that a high proportion of the light-producing radiation is concentrated into a very short spectral range, over which the refractive indices of the refracting media of the eye would show little absolute change; hence the really bright part of the spectrum yields a very sharp image.

\textbf{Resolution of Visual Instruments}

The mere physical resolution of images in spectroscope, telescope, or microscope will not suffice to give an image resolved by the eye
unless (as mentioned above) it is observed under sufficient magnification; and this in turn depends on the nature of the object and the degree of contrast. The resolution actually observed by vision with telescope and microscope often approaches the theoretical limit when sufficient magnification is used with objects of high contrast, partly because the extreme relative sensitiveness of the eye to the yellow-green (as discussed above) minimizes the effects of the residual chromatic aberration of the optical systems, as well as that of the eye itself.

![Graph](image)

**Fig. 92. The “Visibility” or Relative Luminosity Curve.**

**Useful and Empty Magnification**

The magnification of a visual instrument can, then, usefully be increased until the smallest detail physically resolved is seen by the eye under a sufficient angle for comfortable vision. To fix ideas, it can be shown that the formulae for the physical resolving limit derived above for point sources usually apply fairly well to objects such as gratings of equi-spaced parallel bars and spaces if the grating interval (centre to centre of adjacent bars) is substituted for the interval between the sources. The highest contrast is exhibited when the bars are completely black against a bright background; the resolution is then *fairly easily* seen by an emmetropic or corrected

* The grating can often be seen by persons with good eyesight (though with increasing difficulty) until the subtense diminishes to one minute of arc or even less.
eye if the subtense of the interval is about four minutes of arc in the visual field. If the contrast is weak, however, a much larger visual angle may be necessary.

Suppose then that the physical resolving limit of a telescope corresponds to the angular interval \( \omega_0 \) between the elements of the object (double star or grating, etc.). Then, from (4.15),

\[
\omega_0 = \frac{1.22}{A}
\]

If this minimum detail is now magnified so as to subtend 4 minutes, or 15\textsuperscript{th} degree,

\[
M\omega_0 = \frac{1.22\lambda}{A} \quad M = \frac{1}{15} \left( \frac{\pi}{180} \right)
\]

Thus

\[
M = 0.0095 \, (A/\lambda)
\]

If it is desired readily to see the smallest high-contrast detail resolvable by a well-corrected objective, the magnifying power may be usefully given the above value. Giving \( \lambda \) a mean length for visible light, say 0.5 \( \times \) 10\textsuperscript{-4} cm, a useful numerical equation is obtained—

\[
M \simeq 19 \times \text{(diameter of objective in centimetres)}
\]

Hence the value \( A/M \), which will be the diameter of the exit pupil in centimetres, will be 15, or approximately half a millimetre. Increase of the magnifying power beyond this point (by the use of a stronger eyepiece or by an objective of the same aperture but longer focal length) will fail to reveal any more high-contrast detail; the image will increase in size, but will become proportionately more diffuse or "fuzzy."

In the case of the microscope, the numerical value of the angle under which an object of height \( h \) appears if the magnification is \( M \), is \( Mh/b \) (see p. 82). If then \( h \) corresponds to the resolving limit (0.61\( \lambda/NA \)), where \( NA \) is written for the numerical aperture, and the subtense is magnified so as to reach four minutes of arc (as explained above)

\[
\frac{0.61\lambda M}{NA \times b} \simeq \frac{1}{15} \left( \frac{\pi}{180} \right)
\]

Putting \( b = 25 \) cm, and \( \lambda = 0.5 \times 10^{-4} \) cm, we thus obtain (disregarding signs)

\[
M \simeq 954 \, (NA)
\]

The necessary magnification for the above requirement is therefore directly proportional to the Numerical Aperture of the objective.
A value of one thousand times the Numerical Aperture is perhaps easily remembered, and will serve as a useful guide for many purposes.

**Spectacles**

A normal or "emmetropic" human eye, when unaccommodated, forms a sharp image of infinitely distant objects; however, either

![Diagrams showing hypermetropia and myopia.](image)

Fig. 93. Illustrating the correction of hypermetropia and myopia.

the axial length of the eyeball or the curvature of the optical elements at rest may be greater than necessary. The "far-point" (i.e. the object point giving a sharp image) is now at a measurable distance in front of the eye, which is said to be **myopic**. On the other hand the eyeball may be too short for the optical system. Parallel rays of light, from an infinitely distant object, entering such an eye have not reached their focus before they strike the retina; and if there is
to be a sharp image there, the rays must be made slightly convergent before they reach the eye. They can then be conceived as converging towards a (virtual) "far-point" situated behind the eye at a finite distance. The eye is then said to be hypermetropic (or more briefly hyperopic). Fig. 93 shows these conditions.

In each case the far-point is that point which is conjugate to the axial point of the retina, with regard to the refracting system of the eye.

The function of a spectacle lens for distance vision is to project the image of a distant axial object into the far-point. If, therefore, we describe the refractive condition of the eye in terms of the position of the point of accommodation, there is no necessity to consider any of the internal optical actions.

When the accommodation is fully exerted, the eye is focused upon the near-point. Young children have a very great range of accommodation, but, owing to the diminishing elasticity of the lens as age progresses, the range is progressively diminished. If $k$ is the distance of the far-point from the vertex of the cornea, the "vertex refraction" of the eye is the reciprocal of $k$. If $b$ is the distance of the near-point, the amplitude of accommodation is $1/k$ minus $1/b$ (the metre is the proper unit of length). There is also usually a loss of refractive power in persons over 50 years old; so that even the far-point is behind the eye, as in hypermetropia. However, the refractive conditions characteristic of advancing age are known as presbyopia. The following table gives some indication of the usual changes in a normal case—

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Distance of near-point (metres) ($b$)</th>
<th>Distance of far-point (metres) ($k$)</th>
<th>Amplitude of accommodation ($1/k - 1/b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$-0.071$</td>
<td>$\infty$</td>
<td>$14.0$</td>
</tr>
<tr>
<td>20</td>
<td>$-0.10$</td>
<td>$\infty$</td>
<td>$10.0$</td>
</tr>
<tr>
<td>40</td>
<td>$-0.22$</td>
<td>$\infty$</td>
<td>$4.5$</td>
</tr>
<tr>
<td>55</td>
<td>$-0.666$</td>
<td>$4.0$</td>
<td>$1.75$</td>
</tr>
<tr>
<td>70</td>
<td>$+1.0$</td>
<td>$0.80$</td>
<td>$0.25$</td>
</tr>
</tbody>
</table>

A young person whose distance vision is properly corrected by spectacles can focus on near objects by exerting accommodation, but older people who have lost this power must in general have special spectacles for reading, etc., or critical observation at other distances. It may happen, of course, that the distance-vision of a person who is somewhat myopic in youth may improve with advancing age. Note that if accommodation is exerted by a hypermetropic eye, the positive distance of the point of accommodation
tends to increase at first, but may pass (through infinity) to negative values if the amplitude of accommodation is sufficient.

**Aphakic Eye**

If the crystalline lens of an eye becomes opaque through disease, it may be removed by operation. The eye is then styled "aphakic," and the residual refractive power of the optical system is almost wholly due to the high curvature of the corneal surface. If the eye is normal previous to the operation the power of the correcting spectacle lens (for distance vision) to be mounted in front of the aphakic eye is usually about 11 D or 12 D. There is, in such eyes, no appreciable power of accommodation.

**Numerical Example.** Calculate the distance correction required by an aphakic eye, the length of the eye from the cornea to retina being 24 mm, the corneal radius 8.0 mm, and the refractive index of the internal media 4/3. The lens will be worn 12 mm in front of the eye.

The first step is to calculate the position of the object point conjugate to the retina, the optical system consisting solely of the one refracting surface. Referring to equation (1.19), i.e.

\[
\frac{n'}{l'} - \frac{n}{l} = \frac{n' - n}{r}
\]

the data given above indicate that \(l' = 24\), \(n' = 4/3\), \(n = 1\), and \(r = 8\); whence \(l = 72\) mm; that is, 72 mm behind the eye. Thus the point of fixation will be \(72 + 12 = 84\) mm behind the lens. This lens has therefore to project an image of an infinitely distant object into this point and its (positive) focal length is 84 mm. This is 0.084 metres, and the power is the reciprocal, i.e. +11.9 D.*

**Determination of Vertex Refraction**

Suppose that an eye is assisted by a spectacle lens of which the vertex power is known, i.e. the reciprocal \(F_v\) of the distance \(f_v'\) in metres between the posterior surface vertex and the second principal focus. The distance between the latter vertex and the vertex of the cornea is \(d\). Then if \(k\) is the distance between this corneal vertex and the point of accommodation, the condition for sharp vision of distant objects is

\[k = f_v' - d\]

* Owing to the depth of focus of the eye, it is usually sufficient to give results to the nearest one-tenth of a diopter.
Writing $K$ for the reciprocal of $k$, the so-called "vertex refraction" of the eye,

$$K = \frac{1}{f_v' - d} = \frac{F_v}{1 - dF_v} \quad \ldots \quad (5.01)$$

This right-hand member of this equation is sometimes called the "effectivity" of the lens at the distance $d$ from the vertex. Alternatively the relation may be easily put into the form—

$$F_v = \frac{K}{1 + dK} \quad \ldots \quad (5.02)$$

so that once the value of $K$ is known the required value of $F_v$ for use at any new distance $d$ can readily be calculated.

**Axial Astigmatism**

The section of a refracting surface of the eye by a plane containing the axis of symmetry is usually called a meridian. It frequently occurs that some difference of curvature exists between the various meridian sections especially of the cornea, so that there is a difference of power from one meridian to another. This effect (see the discussion in the final chapter of this book) is known as astigmatism.

A theorem due to Euler* shows that at any point of a continuous curved surface the normal or principal meridian sections of maximum and minimum curvature will be perpendicular to each other; and also that this has the following sequel. Let $F_1$ be the power in one of these meridians, and $F_2$ that in the perpendicular one; then the power in a meridian making an angle $\theta$ with the first will be

$$F_{\theta} = F_1 \cos^2 \theta + F_2 \sin^2 \theta$$

The amount of the astigmatism is defined as $F_1 - F_2$. Unless the surface of a cornea is very irregular (so that the above law applies only to infinitesimal areas) the fault can then be corrected by a lens having the necessary compensating difference of power between perpendicular meridians, and so mounted that the effectivity of the lens in the two meridians of the eye agrees with the two vertex refractions at the cornea. In other words, the principal meridians have different far-points, and the corresponding meridians of the correcting lens must project the images of infinitely distant objects into these positions respectively. The lens can, for example, have one spherical and one "cylindrical" surface—or curved surfaces with different powers in different meridians ("toric" surfaces) can also be used. Since Euler's theorem applies also to such surfaces it follows

that if the powers in the principal meridians are corrected, there will be correction in all other meridians (fuller details will be found in works on ophthalmic optics). It may be noted here that most of the elementary problems in connexion with axial astigmatism can be solved by applying the principles of the foregoing sections separately to the principal meridians.

The sign convention for the meridian direction is such that the zero direction is horizontal when the head is upright, and increases in the clockwise direction as seen by the astigmatic eye itself. Alternatively, the meridian angle increases in the anti-clockwise direction as seen by the oculist or optician examining a patient's eye. The meridians are numbered from $0^\circ$ to $180^\circ$ (Fig. 94).

**Numerical Example.** Find the ocular astigmatism (referred to the vertex) of the eye corrected for distance by the following lens—$+4.75\ \text{S} / -2.25\ \text{C Ax. 30}^\circ$, placed 14 mm from the eye. This means that the correcting spectacle lens (considered thin) has one surface spherical (S) of power $+4.75$ diopters and the other surface cylindrical (C) of negative power $2.25$ diopters with axis at $30^\circ$.

A cylinder has no power in the direction of its axis. Hence the power in the principal meridian $30^\circ$ will be $+4.75$ and the corresponding far-point of the eye will be $(1/4.75)$ metres behind the lens, i.e. $0.2106$ m. The far-point is thus $(0.2106 - 0.014) = 0.1966$ m behind the vertex of the eye. The corresponding vertex refraction is $+5.087$ D. Again, the power of the lens in the meridian $120^\circ$ will be $4.75$ minus $2.25 = 2.5$ D; thus the far-point will be $0.40$ m behind the lens; and so $0.386$ m behind the vertex of the eye. The corresponding vertex refraction will be $2.59$ D. The difference in the powers of the principal meridians is thus $2.497$ diopters, and this is the ocular astigmatism of the eye, of which the vertex refractive condition could be described (nearly enough) as

$$5.09\ \text{S} / -2.5\ \text{C Ax. 30}^\circ$$
Alternatively, such results can be obtained by the formal application of equation (5.01).

**Vision with Instruments**

Myopic or hypermetropic eyes need no spectacles with instruments such as telescopes and the like, because the position of the eyepiece with regard to the first image can in general be adjusted in order to project the second image into the point of accommodation. The astigmatic eye cannot, however, obtain a sharp image unless a suitable correcting lens is used to supplement the instrument. The user of an instrument may wear his spectacles; or, if these prevent a sufficiently close approach of the eye to the eyepiece, a small cylindrical lens of the requisite compensatory power can be mounted immediately behind the eye lens of the instrument in the required orientation.

In general, instruments are so designed that the pupil of the observer’s eye is brought into the plane of the exit-pupil of the instrument. If the latter is smaller than the eye pupil, *all* the light transmitted by the instrument can now enter the eye (except for any loss by corneal reflection, etc.) and the illuminated region on the retina corresponds to the full field of view. If the instrument is fixed and also the head, the eye must rotate in its socket in order to get the various parts of the field in distinct focus on the *fovea*, and this results in the obstruction of some of the rays by the edge of the iris (Fig. 95), so that this mode of observation is restricted. However, there is usually some scope for movement of the head with respect to the instrument so that the eye pupil may retain its required position even when the eye is rotated. Moreover, most instruments will allow of some change of direction in use, so that any required object can be brought to the centre of the field.

**Physical Recording Methods**

It is not within the scope of a book on Geometrical Optics to go into the physics of photographic and photo-electric recording methods, except in so far as they have a bearing on some of the questions of the resolving power, etc., of instruments and therefore on instrument design. In the present case only the briefest indication of some leading considerations is possible.

The usual photographic material consists of an emulsion of minute crystals of silver salts (iodide, bromide, etc.) dispersed in a thin layer of a solidified substance such as gelatine spread over glass, or over a flexible film. Many mechanical considerations require that the surface of the film shall be flat while the record is made, and this
itself is a special optical requirement (see Chapter VII). While ordinary plates have a fairly selective action (the most intense photographic effects occur in the blue, violet, and near ultra-violet

region of the spectrum (see Fig. 96), the production of natural-looking photographs demands the use of panchromatic materials in which the response is much more uniform over a very wide range of the

spectrum. The "secondary spectrum" of a lens may thus have a more marked effect on the definition of a photographic image than it has on a visual one. However, the importance of the secondary spectrum depends greatly on the relative spectral energy distribution characteristic of the source (see Fig. 97 for some examples).
Light entering a photograph emulsion is diffused by reflections and scattering at the surfaces of the grains of silver halides, and often also by some back reflections (halation) at the surfaces of the supporting base. Such stray light will weaken the contrast of the image detail, as compared with the intensities which would be found in the aerial image.

There is, however, an effect even more troublesome. The speed of the emulsion is directly associated with the size of the grains, which cannot usually for this reason be made small in relation to the resolving limit of the lens with which the film must be used. The finite sizes of the grains themselves must set a limit to resolution. The grouping of the grains in the emulsion is, moreover, not uniform, and even in a negative of apparently uniform density the examination of the densities of successive equal areas shows an increasing statistical variation as smaller and smaller areas are taken. This causes a "grainy" or non-homogeneous effect in a moderately enlarged photographic image, even though the separate grains remain invisible. The "granularity" is a parameter expressing the statistical variation; it is roughly proportional to the diameter of the grains when these are large. For such reasons the granularity of the material often tends to set a limit to the resolution of detail in the photographic records, and it is found in typical cases that the actual resolving limit distinguishable in the negative exceeds by ten times or more the magnitude which might be expected from the "Airy disc formula." The explanation of the actual performance is a very complicated matter. In spite, however, of the limitation to the

![Graph showing energy distribution for different light sources](image-url)
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resolving power due to the properties of the materials, it does not follow that large aberrations in the optical paths can be allowed without some loss in resolution.

In television systems, etc., the image projected by a lens may be received on a photo-sensitive surface, either continuous or discontinuous, which emits electrons. A scanning system is generally employed; for example the screen may be scanned by a cathode-ray beam brought to a sharp focus thereon. The resolving limits of the system are now affected partly by the structure of the above surface and partly by the diameter of the cathode-ray spots, both in the image converter and in the oscilloscope usually employed as a receiver. In such cases the resolving limit may bear even less relation to the characteristic physical resolving limit of the lens.

EXERCISES V

1. How far must a $-7$ D lens be placed from an eye myopic by $-5$ D, in order that a distant object may be seen distinctly?

2. An eye is hypermetropic by $+6$ D. Find the necessary distance at which a $+3$ D lens must be placed in order to obtain distinct vision of a distant object.

3. What is the vertex refraction and the amplitude of accommodation of a person whose far point when wearing a $+3$ D lens is $33\frac{1}{4}$ cm outside the lens, and whose near point when wearing a $+8$ D lens is $15$ cm outside it? Each lens is worn $15$ mm from the cornea.

4. Prove that the defect in the axial length of an eye is given approximately in millimeters by $x$, where $x = \frac{1}{2} K$, $K$ being the refractive error in diopters. N.B. the two focal lengths of the optical system of the eye can be taken to be $16.5$ mm and $22$ mm, numerically.

5. Show that, if a thin lens in contact with the eye forms an image of the near point at the far point, its power is equal to the amplitude of accommodation.

6. The farthest point of distinct vision is $52$ cm from the eye when a $+10.0$ D lens is placed $20$ mm in front of the cornea. The nearest point is at $12$ cm from the cornea with the same lens. Calculate the refractive error, the amplitude of accommodation, and the position of the near point when the proper distance correction is worn at $12$ mm from the cornea.

7. An eye views a distant object consisting of horizontal and vertical lines. Viewed through a $+3$ D cylindrical lens, with axis horizontal, the vertical lines only are clear. If the axis of the lens is made vertical, the horizontal lines only are clear. There is no apparent difference of clarity between the horizontal and vertical lines when they are viewed through a $+1$ D cyl., axis vertical. What can be inferred about the refraction of the eye, and its amplitude of accommodation?
8. Find the refraction and astigmatism, referred to the vertex, of the eyes corrected by the following lenses, which are mounted 14 mm in front of the cornea in each case—

(a) \( +4.75 \text{ S } / -2.25 \text{ C Ax. 30°} \)
(b) \( -4.50 \text{ S } / +6.50 \text{ C Ax. 135°} \)

9. Give the possible alternative ways (using spherical and cylindrical surfaces only) of providing the corrections given in No. 8 above, assuming in this case that the distance of the spectacle lens from the eye remains unchanged.

10. A double star with equal components subtends an angle of 4 seconds of arc in the sky. What is the minimum magnifying power you would recommend for an astronomical telescope to exhibit it? What is the minimum diameter required for the objective if the two star images are to be seen clearly separated by a perfectly dark interval between them?

11. Show that the radius \( p' \) of the exit pupil of a microscope can be numerically expressed by

\[ p' = NA \cdot \frac{b}{m} \]

where \( NA \) is the Numerical Aperture, \( b \) is the distance of distinct vision (for which the instrument is focused), and \( m \) is the visual magnification. Hence estimate the diameter of the exit pupil when sufficient magnification is used to enable the eye to recognize all the high-contrast detail resolvable by the objective.

12. Given that the angular diameter of the moon is approximately 0.01 radians, and that the anterior focal length of the eye is 15 mm, find the approximate number of foveal cones in the retina covered by the moon’s image; it may be assumed that the cones form a hexagonal mosaic with mutual separation of the centres equal to 2 \( \mu \).
CHAPTER VI

Photometry

The present discussion of the subject of photometry will be largely confined to those aspects connected with optical instruments. The physical basis of the matter is the study of energy flow in electromagnetic radiation in space, and is therefore amenable to treatment by the general theory of vector fields, such as can be applied, for example, to hydrodynamics. In the case of a liquid the subject of theoretical discussion may be the rate of transfer of mass across a given area, whereas in radiation we may consider the rate of transfer of energy, which is a measure of the "intensity" of the field. In hydrodynamics we consider "sources" and "sinks"; in radiation fields we also have sources and absorbing materials; and the lines of energy flow can be identified for most purposes with the rays. There may be curved lines of energy flow if the medium is not homogeneous. However, in the elementary photometry of instruments the media are homogeneous and isotropic, and the interest is so specialized that an ad hoc treatment is more convenient.

The theorems of photometry usually relate to a steady condition of flow, and the energy is then supposed to be evaluated, not absolutely in ergs, etc., as in physics, but in terms of its power to evoke the sensation of light in an average human eye (see below, p. 162). For the present, the discussion will refer to light of fixed spectral composition. A quantity of light, thus evaluated, is termed the "flux," and is denoted by the symbol \( F \); relative amounts of flux are proportional to the relative energy only if the spectral composition is constant.

Standard Concepts of Photometry

Consider an infinitesimal luminous surface element, of area \( ds \), radiating light into a homogeneous medium (Fig. 98); some of this light falls on a distant continuous surface, an element of which, of \( \delta A \), is thus illuminated. If the distance \( r \) between the source and the element is very large in comparison with the maximum linear dimensions of \( \delta A \), all the rays from the infinitesimal source to the boundary of \( \delta A \) will lie in a surface indistinguishable from that of a cone with its base in the element.
Neglecting consideration of diffraction, etc., it can be postulated that any section of such a cone would be traversed by the same quantity $\delta F$ of light from the source. Given that the corresponding illumination $E$ at any point of the receiving surface is verbally defined as the quantity of light, or "light flux," incident per unit area of the surface, we write

$$E = \frac{dF}{dA} \quad \ldots \quad (6.01)$$

with the usual connotation of the differential coefficient.

Since at distances $r_1$ and $r_2$ the normal cross-sections $\delta A_1$ and $\delta A_2$, respectively, of a cone of very small solid angle will be such that

$$\frac{\delta A_2}{\delta A_1} = \left(\frac{r_2}{r_1}\right)^2$$

the illumination levels $E_1$ and $E_2$, respectively, of such normal sections will be such that

$$\frac{E_1}{E_2} = \left(\frac{r_2}{r_1}\right)^2 \quad \ldots \quad (6.02)$$

It may be inferred generally that the illumination received on a small screen placed perpendicular to the light rays derived from a source small in comparison with its distance varies inversely as the square of this distance. This is the "inverse square law."

Again, if the cross-section of the narrow cone at the distance $r$ is taken so that its normal is at an angle $\phi$ to the rays crossing the perpendicular section, the area will now be $\delta A/\cos \phi$ (Fig. 99). Hence the relative illumination $E_{1\phi}$ will be

$$E_{1\phi} = \frac{dF}{dA_1 \cos \phi} \quad \ldots \quad (6.03)$$

This is sometimes called the "cosine law of illumination."

Arguments based on a discussion of an infinitesimal source lead to formulae which can be applied without appreciable error to small finite sources if the distance of action is relatively large.
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The light radiated from a very small but finite element of an emitting surface into various directions is generally a function of direction. A measure of the "illuminating power" $\delta I$ in the direction of a normal element of area $\delta A_1$, as above, is the "flux per unit solid angle," i.e.

$$\delta I = \lim_{\delta A_1 \to 0} \frac{\delta F}{\delta A_1/r_1^2} . . . . \quad (6.04)$$

We must, however, consider the case of a source of appreciable size. If it is imagined that the element considered above is one of a great number ($N$, say) of elementary sources situated near it, and comprising the complete source, each of these will produce a contribution to the illumination of the same receiving-surface area. Lambert (1760) argued that since the rays from such separate sources do not interfere with each other, their effects will be additive; e.g. if two independent sources separately produce equal illumination on $\delta A_1$, the illumination when they act together can be defined as double that due to either of them acting alone; and so on.

If the source is large, the solid angle $\delta \omega$ subtended by a near receiving element may vary at different parts of the source. The light $\delta F_1$ received from one source element, being

$$\delta F_1 = \left( \frac{dF}{d\omega} \right)_1 \delta \omega_1$$

the total light reaching the receiving element from the complete source is then

$$F = \sum_1^N \left( \frac{dF}{d\omega} \right) \delta \omega = \sum_1^N \delta I \cdot \delta \omega . . . . \quad (6.05)$$

where the summation has to be taken for all the elements of which the source is composed; and it has to be remembered that both $\delta I$ and $\delta \omega$ may vary from point to point of the source. However, if the separation between the whole source and the element $\delta A_1$ is so large, in comparison with the size of the source, that the solid angle $\delta \omega$ is constant within the limit of observation, then we can write

$$\frac{F}{\delta \omega} = \sum_1^N \delta I = I . . . . \quad (6.06)$$
In order to obtain a precise specification, the illuminated normal element can be imagined to lie on a spherical surface surrounding the source, the radius of the sphere being so great that the dimensions of the source are relatively negligible. The left-hand side of the equation then represents the "light flux per unit solid angle" being sent to the receiving surface in the special direction. It is the candle-power* or illuminating power \((I)\) of the source in that direction, and is the sum of the contributing candle-powers of all the elements.

![Polar curve of candle-power for a typical electric lamp.](image)

Some sources may have candle-powers which are roughly uniform in various directions; for example some electric lamps distribute light with some approach to symmetry round the axis of symmetry of the globe. "Polar diagrams" (Fig. 100) may then be used to show the candle-power in various directions by the length of the vector lines. The "inverse square law" is not accurate for a source of finite size except at large distances (see p. 167).

So far, the source has merely been considered as a collection of small elements, and this is a convenient supposition if it is, say, an electric lamp with coiled filaments in a zigzag pattern; but sometimes it is a continuous surface; and the concept of the luminance \((B)\) (in former times often called brightness) is then useful.

* The nomenclature of photometry is still under discussion.
The luminance of a surface is the candle-power or illuminating power per unit projected area (the projection being on a plane normal to the direction of radiation). If a self-luminous surface is smooth and structureless—and we shall restrict attention to such cases—the luminance and element candle-power will be symmetrical about the normal, and will be functions of the angle $\theta$ between the normal and the direction of radiation. We accordingly write the luminance as $B(\theta)$, with a similar notation for the candle power per unit area. Thus

$$B(\theta) = \lim_{\delta s \to 0} \left( \frac{\delta I(\theta)}{\delta s \cos \theta} \right) = \frac{1}{\cos \theta} \frac{dI(\theta)}{ds}$$

where $\delta s$ is the area of the emitting surface element.

A simple expression for the effective candle-power of a finite area of the emitting surface in a particular direction can be written down if the whole source and receiving element are separated by a distance so great that the solid angle $\delta \omega$ subtended by the element at any point of the source varies inappreciably; then the equivalent candle-power is

$$\frac{F}{\delta \omega} = \int B(\theta) \cdot \cos \theta \cdot ds$$

the integration being taken over the area of the luminous surface. As above, the normal receiving element is pictured on a spherical surface of radius very great in comparison with the maximum dimensions of the luminous surface.

**Emission of Luminous Surfaces**

If an iron ball is heated uniformly in a fire till it is at a "bright red heat," all parts of the surface appear to be equally luminous. The eye is not very critical in such an observation, but it is worthwhile to consider formally the relation between the luminance and the direction of emission which would produce such an effect. If the surface is structureless, any variation of its luminance at any point will be expected to be symmetrical round the normal thereto.

Let the entrance pupil of the eye (of area $\delta A$) subtend a small solid angle $\delta \omega$ constant within the limits of observation at any point of the emitting surface, where

$$\delta \omega = \delta A/r^2$$

and $r$ is the (relatively great) distance between the surface and the eye (Fig. 101). Let the area of an emitting element be $ds$, and let the luminance be $B(\theta)$ in the direction of the eye, at an angle of $\theta$ with
the normal to the surface; then the quantity of light entering the pupil is $B(\theta) \cos \theta \cdot ds \cdot d\omega$. Some fraction $k$ of this effectively reaches the optically conjugate region of the retina.

The *area* of the retina illuminated is $f^2 \delta \omega_s$, where $\delta \omega_s$ is the solid angle subtended at the nodal point of the eye by the emitting element, and $f$ is the object space focal length of the eye; this area will thus, if $r$ is relatively great, be indistinguishable from

$$\frac{f^2 ds \cos \theta}{r^2}$$

![Fig. 101.](image)

Therefore the illumination $E$ of the retina is given by

$$E = \frac{kB(\theta) \cos \theta \cdot ds \cdot \delta A/r^2}{(f^2 ds \cos \theta)/r^2} = \frac{kB(\theta) \delta A}{f^2}.$$  \hspace{1cm} (6.08)

If the apparent brightness is to be uniform, it is to be expected that the retinal illumination* must be constant at all parts of the image; hence $kB(\theta)$ must be constant, i.e. $B(\theta)$ will be constant, since there is no reason to suppose that $k$ will vary appreciably if the pupil has a fixed size. The candle-power per unit area of the surface, in an oblique direction, which is $B(\theta) \cos \theta$, will then be simply proportional to $\cos \theta$. Lambert inferred this relation from his observation of the apparent uniform brightness† of the sun’s disc; but although it holds approximately for many cases of emitting surfaces, and (to a lesser degree) diffusely reflecting surfaces, it is in no sense a law of Nature. A surface manifesting it would be termed a “perfectly diffusing surface.”

* It will be noted, in the expression for the illumination of the retinal image, that it is proportional to $k \delta A$, where $k$ is an over-all effective transmission coefficient, and $\delta A$ is the area of the pupil. Actually these are found to be not independent of each other (Stiles-Crawford effect), but we need only note that if the area of the pupil effective in forming the retinal image is cut down the illumination of the retinal image is correspondingly reduced.

† Though considerable variations are, however, found if suitable methods of measurement are used, it does not invalidate Lambert’s argument.
Amount of Light Radiated by a Diffusing Surface

Let \(ds\) be a very small area of a diffusely radiating surface surrounded by a spherical surface of relatively large radius \(r\) (Fig. 102). An annular area on the sphere is marked by the intersection with it of cones symmetrical about the normal to \(ds\) and having semi-apical angles \(\theta\) and \(\theta + d\theta\). The annular area is evidently \(r\,d\theta \times 2\pi r\sin \theta\), and the solid angle \(d\omega\) thus marked out is \(2\pi \sin \theta\,d\theta\). The candlepower of the element in the oblique direction being \(B(\theta)\cos \theta\,ds\), the light \(dF\) radiated into \(d\omega\) is

\[
dF = B(\theta) \cos \theta\,ds \cdot 2\pi \sin \theta\,d\theta \quad (6.09)
\]

Hence the total light sent with a cone of semi-apical angle \(\phi\) is

\[
F = 2\pi \int_0^\phi B(\theta) \sin \theta \cos \theta\,d\theta \quad (6.10)
\]

If the luminance \(B(\theta)\) is constant, the expression can be integrated—

\[
F = \pi \int ds B \sin^2 \phi \quad . \quad . \quad . \quad (6.11)
\]

The total amount of light radiated into space on one side of the element is thus \(\pi Bds\).

If the element is not self-luminous but is reflecting light, and the incident flux is \(F_0\), then

\[
F = \rho F_0
\]

where the factor \(\rho\) is the general reflection coefficient.

Most actual diffusely reflecting surfaces show some tendency towards a "specular" or mirror-like reflection in addition to the diffusion. The luminance of such a surface is then a complicated function involving the angles of illumination and radiation as regards altitude and relative azimuth. Expressions like (6.10) are not applicable to such cases.

The Effective Luminance of an Optical Image

Consider the expression (6.09) for the quantity of light radiated by the element \(ds\) into the solid angle between the two cones as defined in the foregoing section, and let the cone be symmetrical about the axis of an optical system which can be supposed to form an image of the source. The rays in the conical surfaces (Fig. 103) being traced through the system will pass through the image, of area \(ds'\), in conical surfaces of semi-angles \(\theta'\) and \(\theta' + d\theta'\) enclosing a solid angle
$2\pi \sin \theta' \, d\theta'$, where the related values of $\theta$ and $\theta'$, etc., are controlled by the optical sine relation (p. 125)

$$nh \sin \theta = n' h' \sin \theta'$$

(6.12)

(where $h$ and $h'$ can be taken as the diameters of an exceedingly small object and image respectively).

If the effective luminance of the image in the direction $\theta'$ is given by $B'(\theta')$ the amount of light in the solid angle defined above must be

$$dF' = B'(\theta')2\pi \sin \theta' \, ds' \cos \theta' \, d\theta'$$

Now since the ray paths through the image are the continuation of those originally derived from the object, the flux $dF'$ would be equal to $dF$ were it not for the losses by reflection, etc., in the instrument.

In general

$$dF' = k(\theta) \, dF$$

where $k(\theta)$ is a fractional coefficient, and is (in general) a function of $\theta$. Thus

$$k(\theta) \, B(\theta) \, 2\pi \sin \theta \, \cos \theta \, d\theta \, ds = B'(\theta')2\pi \sin \theta' \, \cos \theta' \, d\theta' \, ds'$$

(6.13)

Differentiation of (6.12) gives

$$nh \cos \theta \, d\theta = n'h' \cos \theta' \, d\theta'$$

(6.14)

and on multiplying (6.12) and (6.14) we obtain, putting $h^2 = ds$, etc.,

$$n^2 \, ds \sin \theta \, \cos \theta \, d\theta = n'^2 \, ds' \sin \theta' \, \cos \theta' \, d\theta'$$

(6.15)

Thus on dividing (6.13) by (6.15) and rationalizing we find,

$$\frac{k(\theta)B(\theta)}{n^2} = \frac{B'(\theta')}{n'^2}$$

(6.16)
However, if the object were perfectly diffusing, and the transmission \( k \) of the system for various zones were constant also, the image would radiate (within the limits of angular transmission imposed by the system) as if it were a perfectly diffusing surface. If then the image and object are in air so that \( n' = n \), then

\[
B' = kB
\]  

(6.17)

where \( B \) and \( B' \) are the respective luminances of object and image. It will be clear that \( B' \) must always be less than \( B \) owing to the inevitability of losses by reflection and absorption in any actual instrument. In fact it can be shown that no visual optical instrument

![Diagram](https://via.placeholder.com/150)

**Fig. 104.**
(Perspective view.)

can be devised in which the apparent luminance of the image exceeds that of the object, though indeed the discussion given above only concerns a special case.

**Effective Luminance of a Refracting or Reflecting Surface**

A very small element \( ds \) of an emitting surface, situated in a medium of refractive index \( n \), radiates light towards a refracting surface beyond which the refractive index is \( n' \). It is required to find the consequent effective luminance of the refracting surface, the luminance of the emitting surface being known.

Consider a very small area \( \delta A \) of this refracting surface (Fig. 104), at a distance \( r \) (large in comparison with the maximum diameter of either \( ds \) or \( \delta A \)) from the source, where the mean angle of incidence of the rays from \( \delta s \) is \( i \). If the normal to the source element makes an angle \( \theta \) with the main ray, the amount of light sent from \( \delta s \) to \( \delta A \) will be

\[
B(\theta) \cos \theta \cdot \delta s \cdot \delta A \left( \cos i \right) / r^2 = B(\theta) \delta A \cdot \delta \omega \cdot \cos i
\]  

(6.18)
where $B(\theta)$ is the luminance of the element in the direction $\theta$ and $\delta \omega$ is the element of solid angle subtended by $\delta s$ at $\delta A$.

If a fraction $k(i)$ of this light is transmitted it will be radiated into an element $\delta \omega'$ of solid angle containing the refracted rays, of which the mean ray makes an angle of $i'$ with the normal at $\delta A$. Let $B'$ be the effective luminance of the element $\delta A$, then the light sent onward will be

$$B' \delta \omega' \delta A \cos i'$$

which must be equal to the transmitted light; thus

$$B' \delta \omega' \delta A \cos i' = k(i)B(\theta) \delta A \delta \omega \cos i$$

However, a straightforward discussion* (Appendix VII, p. 208) shows that the following relation exists between $\delta \omega$ and $\delta \omega'$—

$$n^2 \delta \omega' \cos i' = n^2 \delta \omega \cos i$$

Hence on dividing (6.19) by (6.20),

$$\frac{B'}{n^2} = k(i) \frac{B(\theta)}{n^2}$$

If now the area $\delta s$ is only a part of a continuous radiating surface, the elements in it contiguous with $\delta s$ will also send light to the area $\delta A$; but the cones of rays from these sources, each with its apex in $\delta A$, will, if $\delta A$ is infinitesimal, separate themselves again after transmission. Thus $\delta A$ will radiate in various directions with luminance characteristic of the various ray paths. Moreover if we consider other areas of the refracting surface contiguous with $\delta A$, each of these will radiate in a similar manner. If, then, the luminance of the original source is constant over the surface, the effective luminance (candle-power, or intensity, per unit area) of the refracting surface will also be constant, but modified from that of the source by the factor $k(i)(n'/n)^2$.

The argument above could have been adapted with little modification to the case of reflection, which, as shown in Chapter II, can be regarded geometrically as a special case of refraction. Reflection coefficients for unpolarized light are discussed below, p. 159. Further, if the radiation from a source of luminance $B$ traverses any number of surfaces with transmission coefficients $k_1, k_2, \ldots, k_p$, separating media of respective refractive indices $n_1$ and $n'_1$, $n_2$ and $n'_2$, we shall find (since $n'_r = n_{r+1}$, etc.) for the effective

* See also Exercise 9, p. 43.
luminance $B'_p$ of the final refracting surface, due to the "transmitted" radiation,

$$B'_p = B \left( \prod_{i=1}^{p} k(i) \right) \left( \frac{n'_p}{n_1} \right)^2 .$$

(6.22)

Where $\Pi k(i)$ represents the product of all the transmission factors $k(i)$; these can be adjusted to cover losses by reflection and absorption in the successive stages. No formal account is taken here, of course, of internal reflections which may in practice modify $B'_p$ if internally reflected light sends contributions to the eye. Such light produces, in general, various images of the source, each with its characteristic position and effective luminance.

The term "effective luminance" implies, of course, the luminance characteristic of the domain of the rays from the source.

If the source is limited and the area of the refracting surface or surfaces is limited, there will be a limitation of the radiation directions. Care must be taken by the beginner to avoid any idea that the refracting surfaces have other properties of self-luminous sources.

**Effective Luminance of Images**

The *ad hoc* investigation of the effective luminance of an image (pp. 153–5) can now be understood as an aspect of a wider investigation. Consider, for example, a small aperture $\delta A$ in an otherwise opaque thin screen near a self-luminous structureless surface of which the luminance $B(\theta)$ is a function of the emission direction. The effective luminance of the aperture will depend upon the direction of the emission of the rays from the original surface. If, however, $B(\theta)$ is constant, the effective luminance of the aperture will also be constant and equal to that of the source.

If an optical system forms an image $\delta s'$ of a luminous element $\delta s$, the arguments of the foregoing section show us how to write an expression for the effective luminance of the last refracting surface (effective, that is within those directions of radiation which pass through the image $\delta s'$). The image $\delta s'$ can now be regarded geometrically as a "hole" emitting the rays which can be received from the last refracting surface; consequently the image itself radiates with the same effective luminance.

The result has now been reached by arguments independent of any consideration of symmetry. We note, however, that the radiation characteristic $B'(\theta)$ of the image must depend upon the emission characteristic of the source, as well as upon the total transmission coefficients characteristic of the paths of the rays.
Effective Luminance of Images in Visual Instruments

The foregoing theorems are sufficient to explain the chief photometric effects in visual instruments. When the eye views a diffusing luminous surface directly, the whole pupil is illuminated, and (6.08) can be used in calculating the illumination of the retinal image.

Now when an instrument is used, the effective luminance of the image is calculable by (6.22); roughly speaking, it is the object luminance multiplied by the total transmission factor of the system. We can again apply (6.08), but the full aperture of the eye pupil may not be used if the exit pupil of the instrument is smaller than this. Thus in a telescope intended for the use at night it is important to design it so that the exit pupil is as large as possible. If it is larger than the eye pupil, the apparent luminance of the image can be made to approach that of the object, except for losses in the instrument. Advantage is then gained because the luminance required for the threshold of vision in the eye is diminished as the subtense of the visual image is increased.

Very small exit pupils are often unavoidable when high magnifications are used in the compound microscope, but the use of a source of high luminance is then usually possible to overcome the difficulty.

Illumination of Images in Optical Instruments

The total illumination of an image area must not be confused in any way with the luminance. On page 154 an expression was obtained for the amount of light $dF$ passing through an image of area $ds'$ and contained between cones of semi-optical angles $\theta'$ and $\theta' + d\theta'$

$$dF = 2\pi \, ds' \, B'(\theta') \sin \theta' \cos \theta' \, d\theta'$$

As on page 153, it follows that if the total cone of rays is symmetrical and the extreme obliquity is $\phi'$ the total light flux per unit area is $E$, where

$$E = 2\pi \int_0^{\phi'} B'(\theta') \sin \theta' \cos \theta' \, d\theta'$$

and if $B'(\theta')$ is constant, and equal to $B'$ then

$$E = \pi B' \sin^2 \phi'$$

Thus while the luminance of an image in a particular direction is independent of the flux passing through it in other directions, all this flux contributes to the "illumination" of the image area.

In a photographic camera, when the aperture of the lens is varied, the illumination of the central region of the plate will be approximately proportional to the area of the effective diaphragm since
this is nearly proportional to \( \sin^2 \phi' \). The "stop number" is defined as

\[
\text{stop number} = \frac{\text{focal length}}{\text{diameter of entrance pupil}}
\]

so that the relative illumination of the image is inversely proportional to the square of the stop number. For example "f/2" would give about sixteen times the illumination expected from "f/8."

**Absorption**

When light is transmitted through a homogeneous transparent substance a certain amount of energy may be absorbed. In a normal case the amount absorbed in an infinitesimal layer perpendicular to the rays will be proportional to the quantity of light and to the thickness of the layer. If \( \alpha \) is the absorption coefficient, the quantity remaining after an incident flux \( F_0 \) has passed through a thickness \( dt \) will be

\[
F_0 - \alpha F_0 dt = F_0 (1 - \alpha dt)
\]

Hence the amount \( F \) remaining after a total thickness \( t \) will be

\[
F = \lim_{dt \to 0} \left\{ F_0 (1 - \alpha dt)^{t/dt} \right\}
\]

Expanding the bracket term by the binomial theorem, we find

\[
F = \lim_{dt \to 0} F_0 \left\{ 1 - \alpha t + \frac{1}{2!} \alpha^2 t (t - dt) - \frac{1}{3!} \alpha^3 t (t - dt) (t - 2dt) + \ldots \right\}
\]

\[
= F_0 \left( 1 - \alpha t + \frac{1}{2!} \alpha^2 t - \frac{1}{3!} \alpha^3 t + \ldots \right)
\]

\[
= F_0 e^{-\alpha t}
\]

**Reflection Coefficient**

The reflection factors \( \rho_1 \) and \( \rho_2 \) of a polished surface of a transparent medium, for light polarized in and perpendicular to the plane of incidence respectively, are given by Fresnel's equations thus—

\[
\rho_1 = \frac{\sin^2 (i - i')}{\sin^2 (i + i')}
\]

\[
\rho_2 = \frac{\tan^2 (i - i')}{\tan^2 (i + i')}
\]

where \( i \) and \( i' \) are, respectively, the angles of incidence and refraction.
The approximate reflection factor for unpolarized white light may be taken as the mean of $\rho_1$ and $\rho_2$.

**Absorption and Reflection by a Plane Parallel Plate**

Let the reflection coefficient of each surface be $\rho$ and the absorption coefficient $\alpha$. Consider a parallel beam incident at a given angle $i$, the angle of refraction being $i'$. The reflection coefficient is then calculable. Let the thickness be $t_0$; then the oblique thickness $t$ along the internal ray is

$$t = t_0 \sec i'$$

If the incident flux is $F$, a fraction $F_0\rho$ is reflected at the first surface, and the amount of light reaching the second surface after transmission is $F_0(1 - \rho)e^{-\alpha t}$. Of this, a portion $F_0(1 - \rho)^2e^{-2\alpha t}$ is transmitted at the second face, while a part $F_0(1 - \rho)e^{-4\alpha t}$ is reflected. The successive amounts emerging at the two faces can thus be written down as follows—

Transmitted components

- $F_0(1 - \rho)^2e^{-3t}$
- $F_0(1 - \rho)^2\rho^2e^{-3\alpha t}$
- $F_0(1 - \rho)^2\rho^4e^{-5\alpha t}$
- etc.

Reflected components

- $F_0\rho$
- $F_0(1 - \rho)^2\rho^2e^{-2\alpha t}$
- $F_0(1 - \rho)^2\rho^4e^{-4\alpha t}$
- etc.

Sum: $F_0(1 - \rho)^2 \frac{e^{-\alpha t}}{(1 - \rho^2e^{-2\alpha t})}$

(Note that the summation only involves a simple geometrical progression in each case.)

If the absorption of the plate is negligible, the summed amounts are—

Transmitted light:

$$F_0 \left( \frac{1 - \rho}{1 + \rho} \right)$$

Reflected light:

$$F_0 \left( \frac{2\rho}{1 + \rho} \right)$$

The value of the reflection coefficient for a surface of crown glass is about 0.04 for normal incidence. The absorption coefficient for optical glass of good quality (centimetre unit of length) should not exceed about 0.008.

The approximate treatment just given will suffice in the case of white light incident on a fairly thick plate. If the plate is thin or the light has a single frequency, the phase relations of the above

From the above results it will appear that in a case when the above equations are valid, if a source of luminance $B$ is seen through such a plate, the apparent luminance will be $B(1 - \rho)/(1 + \rho)$. If the image is seen by reflected light, the luminance will be $2B\rho/(1 + \rho)$. Naturally the complete discussion of the effective transmission of an optical system, or even of a single lens, involves more complex considerations, but the principles already put forward will enable approximate estimates to be made.

**Projection of Light**

A search-light or similar instrument for the projection of light is often arranged so as to project an enlarged real image of a source of high luminance into the region to be illuminated. A simple case from the theoretical standpoint is that of a source of uniform luminance. Taking a point $P$ (Fig. 105) in the distant illuminated region, and tracing imaginary rays back from it into the projection system toward the source, it will be characteristic of a system free from aberration that *all* such rays will intersect the source, apart from any which may be screened off by the source and its auxiliary parts in a reflecting system. Then the effective luminance, at $P$, of the aperture of the projection system will be given by equation (6.22). In case of a mirror searchlight, the "transmission" factor will have to take account of the *reflection* coefficient of the surface and the atmospheric absorption.

If the whole aperture of the mirror appears to an eye at $P$ to be illuminated, the phenomenon is described as a "complete flash," and the candle-power $I$ of the source is given by

$$ I = \text{effective area of the projector} \times \text{effective luminance} $$

The illumination at $P$ will thus be $I/d^2$ where $d$ is the distance the mirror measured from $P$. Note that the illumination at $P$ cannot be increased by enlarging the *size of the source*—but only by using
a source of greater effective luminance, or a system of greater effective aperture.

**Heterochromatic Photometry**

So far, the discussion of photometry has referred to light of fixed spectral composition, and the luminance of a source is then (other conditions being unchanged) directly proportional to its output of energy.

Consider, however, two uniform monochromatic sources of light subtending approximately equal solid angles \( \omega_1 \) and \( \omega_2 \), which can be measured, near the centre of the visual field of an eye. They are of differing colours \( C_1 \) and \( C_2 \). By means of suitable physical instruments (thermopile, etc.) it is possible in principle to measure the energy sent by each of them to the pupil of the observer's eye. Let these energy-amounts be \( E_1 \) and \( E_2 \) in some standard condition, then the energy density per unit solid angle is \( E_1/\omega_1 \) and \( E_2/\omega_2 \). These sources might consist of small diffusely reflecting screens independently illuminated by differing self-luminous sources. If the distance of one of these latter sources is changed, the luminance of the corresponding screen is changed, and increases or diminishes with the relative energy which it reflects.

Now any observer will generally agree that there is an effect—colloquially called the apparent brightness—which can be recognized independently of the differing colour. It is found that, though at first it is difficult for an observer to tell when contiguous fields of differing colours match in their "brightness," his capacity is developed by practice; there are moreover other methods (flicker, etc., described in books on photometry) of achieving much the same end. We may then suppose that the concept of relative luminance has a meaning even in the presence of difference of colour.

Let the relative luminances of the two above screens in the standard conditions be \((E_1/\omega_1)V_1\), and \((E_2/\omega_2)V_2\), where the new unknown factors \( V_1 \) and \( V_2 \) are still to be determined. If these apparent luminances can be brought to apparent equality by quantitative photometric changes, modifying them by factors \( f_1 \) and \( f_2 \) respectively, then the \( V \)-factors are such that we shall have

\[
\left( \frac{E_1}{\omega_1} V_1 \right) f_1 = \left( \frac{E_2}{\omega_2} V_2 \right) f_2
\]

or

\[
\frac{V_1}{V_2} = \frac{E_2 f_2 \omega_1}{E_1 f_1 \omega_2}
\]

so that the ratio of \( V_1 \) to \( V_2 \) can be found, since all the quantities on the right-hand side of the equation are experimental data. In this
way, any number of such sources can be compared. The $V$-factors are called the “visibility” or luminosity factors.* 

Suppose for example that $f_1$ is $\frac{1}{2}$ while $f_2$ is unity. Then we can say that the original luminance of screen 1 is double that of screen 2. This quite independent of any subjective estimate of relative brightness or luminance, apart from the recognition of equality.

Recalling the general principles of photometry explained so far, we know that it is readily possible by means of transparent reflectors, etc., to illuminate a portion of the visual field by light consisting of the sum of a number of independent contributions having the same spectral character; and if the luminance of the contributions reaching the eye are $L_1$, $L_2$, $L_3$, etc., it is a postulate of photometry that the resultant luminance $L$ of the sum of the contributions of $n$ parts is

$$L = L_1 + L_2 + L_3 + \ldots = \sum L_i$$

In a similar way, it is conceived that it is possible to add the luminance contributions of heterochromatic components; although indeed the justification therefor lies largely in the measure of practical agreement of observations with theoretical predictions based on the above assumptions. Thus if $\varepsilon_1$ and $\varepsilon_2$ are the energy densities corresponding to wavelengths $\lambda_1$ and $\lambda_2$, and $V_1$ and $V_2$ are the relative visibility factors, the total relative luminance of the sum is supposed to be

$$L = \varepsilon_1 V_1 + \varepsilon_2 V_2$$

and if there is a continuous series of spectral components added in one field, the energy density per unit range of wavelengths at wavelength $\lambda$ being $\varepsilon_\lambda$, the total relative luminance $L$ of the addition of all the components will be

$$L = \int_{\lambda_1}^{\lambda_2} \varepsilon_\lambda V_\lambda d\lambda$$

Then if the light is passed through a selective light filter with an effective transmission $t_\lambda$ which is a function of wavelength, the total relative luminance of the transmitted light $T$ will be

$$T = \int_{\lambda_1}^{\lambda_2} \varepsilon_\lambda t_\lambda V_\lambda d\lambda$$

and the total transmission factor of the filter will be $T/L$.

**Standards and Practical Units**

The standard source is a full or cavity radiator (black-body radiator) at the temperature of solidification of platinum (2,046°K). The unit of

* The terminology is still under discussion. See the Report of the British Standards Institution, B.S. 233: 1953.
candle-power is such that the luminance of the standard source is 60 candles per square centimetre.

The lumen is the flux radiated within unit solid angle by a uniform source having a candle-power of unity.

The unit of illumination is one lumen per unit area.

Various practical units result from the use of metric and non-metric units of length, e.g. the centimetre and the foot. Thus an illumination of one “foot-candle” is obtained when the inner surface of a sphere of radius one foot is illuminated by a very small source of one candle-power at its centre, radiating uniformly in all directions; a flux of one lumen is then received upon an area of one square foot of this surface.

A unit of luminance which often finds use in visual research is “one candle per square centimetre.” This has been variously called the “stilb” or the “nit.”

**Technical Photometry**

Some of the main subjects of interest in technical photometry are: the measurement of the candle-powers or intensities of sources and the distribution of the flux from them, as well as their total output of light; also the measurement of illumination from existing arrangements of sources. The subject is altogether too wide to make it possible to do more in the present discussion than to give a few notes on some of the methods and problems encountered.

Instruments and methods employed until recent times were largely dependent on visual observations involving the recognition of the equality of luminance of contiguous visual fields. However, the introduction of photo-electric cells has increased the range of methods of measurement; though it is often the case that the use of electrical circuits is attended by characteristic troubles of maintenance, calibration and the like. These may be negligible for rough measurements, e.g. the approximate measurement of illumination with photocell and galvanometer, but when optimum accuracy is essential an elaborate procedure is unavoidable. Hence the older visual methods are still of use for casual observations, but tend to be superseded in standardizing laboratories. Some visual methods and instruments will be described in the first place.

**Practical Sources**

The “standard candle” used long ago has given place to electric lamps operated under prescribed conditions of current and voltage; these are secondary standards, calibrated against the primary platinum standard in a standardizing laboratory. Such lamps usually
have an array of filaments contained in one plane perpendicular to direction in which the light should proceed.

**Use of Photometer Bench and Comparison Head**

The lamp for test and a standard lamp are mounted in sliding carriages on the photometer bench (Fig. 106), and, if visual methods are to be employed, a photometer head is mounted between them. This is an optical device for viewing simultaneously both sides of an opaque screen and thus allowing the visual recognition of equality of luminance. It is illustrated in Fig. 107. Light from the lamps to

be compared illuminates the respective sides of the plate $W$, coated with white diffusing surfaces, from which rays diffused obliquely are reflected by prisms $P_1$ and $P_2$ to meet in the comparison prism $C$, consisting of two 90° prisms with their hypotenuse faces in optical contact. Parts of the surface of one prism are removed by sandblasting before they are joined (either by cementing or otherwise)
so closely that no reflection can take place where the polished faces meet; but on the other hand, total reflection can take place where the air-gap due to the sand-blasting is encountered. Thus the rays entering the lens $L$ are derived from $P_1$ for the outer, and from $P_2$ for the inner parts of the field; hence the light in these separate parts comes from the respective sources on opposite sides of $S$.

The lens $L$ is mounted with a diaphragm $T$ fixed in its focal plane, pierced by a small axial circular aperture (a "stop") a few millimetres in diameter. Hence the rays transmitted by $T$ are those which, before entering $L$, had a direction close to the axis of $L$ and thus were represented by narrow bundles of rays from each side of $W$; all those derived from one side having nearly the same direction. The eye receives the light transmitted by $T$.

The system of lens $L$ and diaphragm $T$ slides in a tube so that the distance of $L$ from the separating surface in $C$ can be varied; if the distance is again equal to the focal length of $L$, the unaccommodated eye looking through $T$ sees the field of separation in sharp focus. The apparent luminance of the two parts of the field would now, in a perfectly constructed system, be proportional to the illumination of the respective sides of $W$. In practice, perfect symmetry is hard to achieve and the measurements have therefore to include some with the photometer head reversed, etc. Reference must be made to text-books on photometry for further details.

Returning to the subject of the use of the bench, the relative distances of the lamps are varied until the illumination is found to be the same on each side. Then if both the sources were extremely small and the lamps of candle-powers $I_1$ and $I_2$ were distant from the screen by $d_1$ and $d_2$ units respectively, we should have

$$\frac{I_1}{d_1^2} = \frac{I_2}{d_2^2} \quad \ldots \quad \ldots \quad \ldots \quad (6.23)$$

Hence if $I_2$ is the standard and known by definition, $I_1$ be calculated with the aid of the experimentally determined $d_1$ and $d_2$.

However, the finite sizes of the sources makes the inverse square law inaccurate if the distances are not relatively large. To illustrate the kind of correction required, let us calculate the illumination due
to a plane circular luminous disc, perfectly diffusing with luminance $B$ at a distance $l$ on its axis of symmetry (Fig. 108). The light $dF$ sent from a symmetrical ring of breadth $dr$ and radius $r$ to an elementary area $ds$ perpendicular to the axis is readily seen to be

$$dF = \frac{2\pi B r\, dr\, ds \cos^2 \theta}{l^2 + r^2}$$

and the total light is then

$$F = 2\pi B\, ds\, l^2 \int_{r=0}^{r=R} \frac{r\, dr}{(l^2 + r^2)^2}$$

The illumination $E$ is then

$$E = \pi B \left( \frac{R^2}{l^2 + R^2} \right) = \frac{\pi R^2 B}{l^2} \left( \frac{l^2}{l^2 + R^2} \right) \quad (6.24)$$

The factor $\pi R^2 B/l^2$ represents in a sense the application of the inverse square law; the second factor represents the modification due to the finite size of the source. Note that when $l$ is very small in comparison with $R$ the whole expression approximates to $\pi B$.

The form of the correction will vary, of course, with other forms of source. Photometer benches are usually very long, so that when $l$ is very large in comparison with $R$, the correction factor may approximate sufficiently nearly to unity.

**Distribution of Flux Round a Source**

If a lamp is so mounted near one end of a photometer bench so that it can be rotated into any angle of "altitude and azimuth" with respect to some fixed point in itself, the apparent candle-power of the source at some fixed distance from the centre of rotation can be determined by standard methods. In this way polar diagrams of the candle-power in various directions can be drawn. If the lamp is symmetrical about an axis the polar diagrams will be identical in any plane containing the axis, but this is often not the case.

The total output of light from the lamp can be estimated by numerical integration over the whole solid angle of the space surrounding the lamp; but this is a lengthy process involving much trouble and experimental inaccuracy. Reference should be made to text-books on photometry for a discussion of the use of "spherical integrators" (Ulbricht spheres), by which the difficulties of the measurement of the total output of light can be largely overcome.

**Measurement of Illumination**

In one sense, the system of the photometer bench with a fixed comparison head and the comparison lamp movable to various
distances can be regarded as a system for determining the illumination of the side of the photometer screen exposed to light from the test source. But in order to be able to measure the illumination at any position, say on a table in a room, a portable system is convenient. One such arrangement is shown in Fig. 109, where the comparison prism C, the lens L, and the telecentric stop T have similar functions to those described above, but are small in size and mounted in a special instrument.

![Fig. 109. Illumination Photometer (Macbeth type).](image)

One part of the field is illuminated by rays received directly from the surface whose illumination is to be measured, the other part by a diffusing glass G screened from external light and illuminated only by light derived from a small lamp E which moves in a tube so that its distance from G can be varied over a relatively wide range. In this way the apparent luminance of the corresponding side of the photometer field is roughly inversely proportional to the square of the distance between E and G.

Annular screens (S) have to be inserted in the tube to catch the "stray light" which might otherwise reach G from E through reflection at the internal walls of the tube.

When the instrument is to be used, a special disc having a white diffusing surface is placed in the position for which the illumination is to be measured, and the viewing axis of the instrument is directed
towards it. Both parts of the matching field are now illuminated, and their apparent luminance is equalized by movement of E. A scale attached to E can be calibrated to read directly in any suitable units; for example, the foot-candles of illumination on the special standard disc. In the calibration, known illuminations can be independently produced on the disc by the use of standard sources at measured distances. It is improbable that the disc will have a perfectly diffusing surface, and the direction of viewing with the instrument has to be properly arranged. Practical precautions and the full details of the methods are discussed in text-books on photometry.

The illumination required for efficiency of vision and tasks dependent thereon has been the subject of much study in recent years and illumination measurements (as well as the selection and disposition of sources) have become of great importance.

Physical Photometry

The principal classes of physical detectors of radiation are—

1. Thermopiles.
2. Bolometers.
4. Vacuum cells.
5. Gas-filled cells.

While thermo-piles and bolometers can be used effectively to measure total radiation, special precautions have to be taken (light filters, etc.) with any of the above detectors if their response (relative spectral sensitivity) is to be related to the response of the human eye. However, some types of vacuum and gas-filled cells, and especially barrier-layer cells, are readily adaptable for photometry in the accepted sense.

Vacuum cells often take the form of a small evacuated bulb of glass coated on the interior with a film of one of the alkali metals. Admission of light through a gap in the film causes the emission of electrons. One electrode (the cathode) is in contact with the film, and another consists of an independent interior plate or wire connected through the glass of the cell; this plate constitutes the anode. The application of an adequate potential difference sets up a current corresponding to the number of electrons emitted. Thus under suitable conditions the current has a linear relation to the illumination of the cell. In gas-filled cells the arrangement is similar, but the current may be greatly enhanced by ionization processes in
the gas, and the linearity may be not nearly so dependable. The currents with weak illuminations are often too small to be measured directly and valve amplification circuits are then employed.

Barrier-layer cells may consist of a thin layer of selenium coated on one side by a transparent layer of a metal such as gold, and on the other side by a supporting metal plate. The action of light incident on the selenium through the gold sets up a potential difference between the two coatings, and the cell thus acts as a weak battery which, when connected through a low-resistance galvanometer, gives a current having an approximately linear relation to the illumination; but the relation may exhibit marked hysteresis, that is to say it depends more or less upon the history of the changes of illumination, current taken, etc., in the period preceding the observation. For some purposes (exposure meters, etc.), the hysteresis may not be important enough to matter, but accurate photometry may require great precautions. Even if the cell is to do no more than is expected of the human eye, i.e. to indicate the equality of some conditions of illumination, precautions are obviously necessary. Cells cannot at present be made in duplicate so precisely that they can be trusted to have exactly equal responses under the same physical conditions.

**Directional Characteristics.** Photo-cells show, in general, marked changes in response according to the direction in which light is incident in or on the cell. Thus, if quantitative comparisons are to be made, precautions must be taken to ensure that the flux is always incident on the cell in precisely the same manner, or so nearly that the variations in the mode of incidence are known to be insignificant. Thus, for example, the light illuminating a surface might be diffused by a suitable coating. If the diffusion were perfect, the light entering a cell mounted in a fixed position from a limited area of the surface would bear a constant relation to the illumination. In practice, no material would give completely perfect diffusion, but such an arrangement might serve provided that the variations of direction of the incident light on the diffusing surface were sufficiently small. Diffusing windows, as of opal glass, may be made to serve similar purposes. The exact arrangements have almost always to be set up according to the special circumstances, and no general rules are likely to be useful. Reference should be made to modern treatises on photometry.
PHOTOMETRY

EXERCISES VI

1. A source of light radiating symmetrically around an axis has a candle-power represented by the product $C \theta$, where $\theta$ is the acute angle between the axis and the direction of radiation and $C$ is a constant. Find the total emission of light from the source.

2. Find an expression for the illumination produced at various parts of a plane screen by a uniformly radiating source of candle-power $I$ held at a perpendicular distance $d$ from it.

3. Two small lamps, each having a uniform candle-power of 10 are hung at a distance of 10 ft above a flat floor at a distance of 10 ft apart. Calculate and plot a curve showing the illumination on the floor along the straight line joining the feet of the perpendiculars from the lamps.

4. Find an expression for the illumination produced by a straight wire filament of length $2l$, but negligible width, at various distances along the line bisecting the filament at right angles on surfaces normal to this line. The filament has a candle power of $C$ per unit length in the normal direction, and is perfectly diffusing.*

5. A semi-infinite medium bounded by a plane contains a very great number of infinitesimal particles in uniform dispersion. Each of them radiates light uniformly in all directions. The effective absorption coefficient of the medium as a whole is $\chi$. Show that the surface of the substance appears to radiate in accordance with the cosine law of emission (or as a perfect diffuser). "Scattering" need not be considered.

6. A plane parallel plate of glass of thickness 2 cm has polished surfaces, each with a reflection coefficient of 0.04; and has an absorption coefficient of 0.20. It screens a luminous diffusing surface of which the original luminance is 1 candle per square centimetre; find the apparent luminance as seen through the glass.

7. A small luminous disc of area $ds$ is perfectly diffusing; it illuminates a plane surface parallel to itself at a relatively large distance. If a zone of this surface is defined by the angle $\theta$ between the direction towards a point in the zone and the normal from the disc to the surface, find an expression for the illumination as a function of the angle $\theta$. Hence comment on the illumination of a photographic plate by a lens in a camera.

8. A plane parallel beam of light enters the entrance pupil of a telescope where the consequent illumination is $I$. If the angular magnification of the instrument is $M$, and the transmission factor is $t$, find the resulting illumination in the exit pupil. Comment on the result, especially with regard to the luminance of the visual image in the telescope.

9. Assuming that a properly focused telescope increases the brightness, but not the apparent size, of the image of a star, and moreover that the same minimum quantity of light is always needed to provide a "just recognizable" image, compare the maximum possible distances of stars of the same candle-power just recognizable with telescopes having object-glasses of diameter $d$.

* An integral which may be useful, but which is not always quoted in textbooks, is

$$\int \frac{dx}{(1 + x^2)^n} = \frac{1}{n} \left[ \frac{x}{1 + x^2} + \arctan x \right]$$
and $d_s$ respectively. It is taken that the effective transmission factors are equal and that there is no obstruction of light by the eye-pupil.

10. A source of light sends a quantity of light ($L$ lumens) to a very small area $ds$ of a perfectly diffusing surface of which the general reflection coefficient is $\rho$. Give an expression for the apparent candle-power of this element in a direction at an angle of $\theta$ with the normal to the surface. Give also an expression for the luminance of the diffusing element.

11. An infinitesimal source of light, of which the total output of flux is $F$, is enclosed within a hollow sphere of radius $r$, of which the walls are perfectly diffusing and have a general reflection coefficient $\rho$. Find an expression for the illumination of any point of the interior due to light which has suffered one or more reflections.

*Hint.* Consider two elementary areas $ds_1$ and $ds_2$ on the interior surface, and write down an expression for the light received by $ds_1$ (say $dF$); hence an expression for the light to $ds_2$. Integrate over the whole surface of the sphere to find the total light received by $ds_2$ after one reflection. Now consider another area $ds_3$ and find the light it sends to $ds_3$ after two reflections; and so on.
CHAPTER VII

Aberrations of Optical Images

It has been shown (pp. 14 and 112) that a bundle of parallel rays can be brought to a point focus by refraction at a refracting surface having the shape of an ellipsoid of revolution of which the major axis is parallel to the incident rays and of which the eccentricity is related to the relevant refractive index ratio. Refraction at a spherical surface having the same axis and paraxial radius of curvature, etc., produces an imperfect union of the rays. It may be alternatively stated that the disturbances derived from a plane wave front perpendicular to the axis of symmetry will meet in the same phase at the focus after refraction by the ellipsoid; but when the spherical surface is in action the phases of the disturbances near the focus will vary. Equation (1.11) represents the gap between the two surfaces as a function of even powers of the radial distance $y$, starting with a term in $y^4$. It would appear reasonable to expect, in the case of a spherical surface, that the optical path-differences of disturbances arriving after refraction at the paraxial focus from various zones will be expressible as a similar series in $y$, starting with a term in $y^4$. However, the concept must be formulated more exactly. Suppose that the rays from an axial object point $B$ are refracted by an axially symmetrical system, and that they are traced through to the neighbourhood of the paraxial image point $B'_p$ (Fig. 110). The locus of points of equal optical path marked off along each ray from $B$ is the orthotomic surface $S$ (Malus) which is identified with the refracted wave front. It is convenient to deal with the wave surface just emerging from the optical system, and to restrict attention to cases in which the focal distance is large in proportion to the aperture.
The aberration characteristic of a ray can be defined geometrically in various ways; for example (Fig. 111), (a) as the longitudinal or axial intercept between the paraxial focus $B_p'$ and the axial crossing point $B_p$ of the ray, or (b) as the transverse intercept $B_p'D$ of the ray in the paraxial focal plane.

On the other hand, the aberration can be defined in terms of optical path. With centre $B_p'$ (Fig. 111), draw the spherical surface AR of radius $l'$, which is intersected at R by the ray which meets the wave surface in P. The path differences of disturbances derived, on to Huygens's principle, from points P and arriving at $B_p'$ will be closely proportional to the width of the gap RP. Thus the aberration $W$ characteristic of the ray path can be defined as the optical path (see p. 110), where

$$W = n'(RP) \quad . \quad . \quad (7.01)$$

The sign of $W$ will be taken to be positive if the wave surface lies in front of the reference sphere, and the corresponding geometrical aberrations will be positive also, as in Fig. 111.

It will be evident from a discussion similar to that on p. 20 that if the axially symmetrical wave surface and the spherical reference surface have the same paraxial radius of curvature, the width $G$ of the gap RH parallel to the axis will be expressible by a series in even powers of $y$ (the height of the point R above the axis) starting with the fourth power.† Hence

$$G = a_2 y^4 + a_4 y^6 + O(y^8) \quad . \quad . \quad (7.02)$$

The oblique gap along the ray is given very closely by

$$RP \simeq G \cos U' \quad . \quad . \quad (7.03)$$

* For Huygens's principle see text-books of Physical Optics.

† However, if the axial point of reference is changed so that the paraxial radii of curvature of the wave surface and the reference surface are no longer identical, the series will be changed, and will start with a term in $y^2$. 
where $U'$ is the angle of slope of the ray, and

$$\sin U' \simeq y/l'$$

so that* $\cos U' = (1 - \sin^2 U')^{1/2} = 1 - \frac{y^2}{2l'^2} + O(y^4)$

(7.04)

Substituting from (7.04) in (7.03) and (7.01) we thus obtain—

$$W = n'(RP) = n' \left[ a_4 y^4 + \left( a_6 - \frac{a_4}{2l'^2} \right) y^6 + O(y^8) \right].$$

(7.05)

It is thus established that, in general, the series in $y$ representing the optical path differences of zonal disturbances arriving in the paraxial focus of an axially symmetrical wave will start with a term in $y^4$. This is called the primary spherical aberration term; the term in $y^6$ is the secondary aberration, and so on.

**Oblique Aberrations**

It is possible, as shown by Sir William Rowan Hamilton in 1828–33, to forecast the type of series involving two variables which must express the aberration even when the object point does not lie on the axis.

Consider a symmetrical optical system with an object point $B_1$ at a distance $h$ from the axis. The plane containing the axis and the object point is an important plane of symmetry. Fig. 112 shows $B'B_1'$ the intersection of the latter with the image plane perpendicular to the axis and conjugate to the object plane in the paraxial

* After the establishment of (7.12), p. 181, the student can show that $\sin U' = (y/l') + O(y^3)$. 
sense. $B_1'$ is the point in the above plane of symmetry which would represent the image point according to the paraxial laws; its height is $h'$.

The point $E'$ represents the axial point of the (paraxial) exit pupil, and the diagram shows a reference sphere with centre $B_1'$ and radius $B_1'E'$. Note that the reference sphere will follow any change of the position of $B_1'$ due to a change of $h$ (and $h'$).

$R$ is a point in the reference sphere; while RA ($=\rho$) is perpendicular to the straight line $E'AB_1'$, and makes an angle $\phi$ with the plane of symmetry; thus $\rho$ and $\phi$ define the position of $R$ in the reference sphere. The related Cartesian coordinates are such that the $X$-axis is the line $AB_1'$, and the $XY$-plane contains the radial direction $h'$. If a "wave surface" corresponding to a disturbance derived from $B_1$ be drawn to pass through $E'$ it will not in general be spherical or centred truly in $B_1'$; but any gap $W$ between wave and reference surface (the gap being defined as explained in connexion with Fig. 111) can only depend on $h'$, and $\rho$, and on $\phi$, for a given pair of conjugate planes.

However, since $W$ must be symmetrical with respect to our main plane of symmetry, it will be unaltered by a change in the sign of $\phi$. Thus the cosine suggests itself as a simple function of $\phi$ to be used in any expression for $W$.

Again when $h'$ is zero the aberration function must be independent of $\phi$ and moreover a function expressible in even powers of $\rho$. This could be true if $h'\cos\phi$ and $\rho^2$ were among the primary forms in which the variables occur in $W$; but the variation of the function $W$ with $\phi$ must cease when $\rho$ is zero. This will be secured if the primary form containing $\phi$ is $\rho h'\cos\phi$. However, while keeping $\rho$ constant the aberration function will be unchanged if the object and image points rotate through $180^\circ$ with respect to the axis and $\phi$ is replaced by $\pi \pm \phi$; the cosine term then changes sign with $h'$ so that $\rho h'\cos\phi$ remains constant as required, but the other terms in $h'$ will change unless they are only present as even powers. For all these reasons, then, the primary forms of the variables necessary and sufficient in the function are

$$h'^2, \rho^2, \text{ and } h'\rho \cos\phi$$

and the complete aberration function is therefore expressible as the most general power series in these variables, provided that the function as well as its differential coefficients is finite and continuous (within the given limits of the variables) and that the series is correspondingly convergent.*

Following a modern notation (see below), such a general expansion can be written in the form—

\[
W = C_0 + \left( C_{20} \rho^2 + C_{11} h' \rho \cos \phi + C_{20} h'^2 \rho^2 \cos \phi \right) + \left( C_{40} \rho^4 + C_{31} h' \rho^3 \cos \phi + C_{22} h'^2 \rho^2 \cos^2 \phi + C_{20} h'^2 \rho^2 \right) + \left( C_{11} h' \rho \cos \phi + C_{00} h'^4 \right) + \ldots
\]  

(7.06)

in which it will be seen that the coefficients \( C \) have subscripts corresponding to the respective powers of \( h', \rho, \) and \( \cos \phi \) appearing in the corresponding terms—merely for the sake of easy identification.

Some of these terms must, however, be zero in the case where the wave surface is taken as coincident with the reference sphere at \( E' \), and when \( \rho = 0 \). This means—

\[
W_{(\rho=0)} = 0 = C_0 + 2C_{20} h'^2 + 4C_{00} h'^4 + \ldots
\]

Thus all these terms will be separately zero in our case and can be omitted from the expression.

Again the aberration of the axial image is now—

\[
W_{(h'=0)} = C_{20} \rho^2 + C_{30} \rho^4 + \ldots
\]  

(7.07)

This is exactly equivalent to the series (7.05) for the spherical aberration (p. 175). As before, we see that if the paraxial focus is chosen, the term in \( \rho^2 \) will disappear; thus \( C_{20} = 0 \) for the paraxial focal plane.

Again let us suppose that the value of \( h' \) is so small that \( h'^2 \) is inappreciable, and that while \( \rho^2 \) is appreciable (paraxial region) terms like \( \rho^4, h' \rho^3, h'^2 \rho^2, \) etc., are negligible; then the residual aberration with all the above limitations will be

\[
W = C_{11} h' \rho \cos \phi
\]

However, referring back to our discussion of the sine relation equation (4.18) (p. 124), this expression is seen to be proportional to the difference of path introduced between disturbances derived from an annular ring (radius \( \rho \)) of the reference sphere by a small shift in the radial direction defined by the plane of symmetry. The term will be zero, by definition, at the image point defined by the paraxial relations, and thus must be omitted from the expression valid for the paraxial-type conjugate point.

The aberration function is now written—

\[
W = (C_{40} \rho^4 + C_{31} h' \rho^3 \cos \phi + C_{22} h'^2 \rho^2 \cos^2 \phi + C_{20} h'^2 \rho^2 \right) + \left( C_{11} h' \rho \cos \phi \right) + (\text{terms of higher order})
\]  

(7.08)
The terms of lower order tend to predominate as the aperture and image height diminish. The expression represents, let us repeat, the optical path differences found at that point which would be assigned as the focus by the paraxial relations. An expression of a related kind was published (though in a different form referring to geometrical ray intercepts) by L. Seidel (1853–6), and the five corresponding terms are often called the "primary" or "Seidel" aberration terms; the first two are called the spherical aberration and coma respectively; the next pair when properly combined represent astigmatism and curvature of field, and the last is the distortion.

If the aberration function is known, it is possible by the application of Huygens's principle, using suitable numerical integrations, to calculate the distribution of light in the region of the focus. When the aberrations are small, so that path differences are of the order of one wavelength or less, this is the only reliable method of forecasting the appearances in the image. The procedure is, however, often complicated and lies outside the possible scope of this book. The phenomenon of the Airy disc has been briefly described (p. 118) and the modifications due to aberration are thus calculated.

When the aberrations are large, reaching amounts exceeding ten or twenty wavelengths, the distribution of light in the image plane has more resemblance to the figure given by the ray density due to the geometrical ray aberrations; and the likeness increases with the amount of the aberrations. It is in this sense that the discussion of ray aberration still has interest. Tolerances for aberrations in optical systems are better stated in terms of optical path when the aberration is small.

**Effects in Other Focal Planes**

A complete study of the aberrations requires discussion of the effects in focal planes other than the paraxial one. From (7.07), representing the case of spherical aberration, it is at once obvious, for example, that the value of \( C_2 \) (which is proportional to the axial change of focus) can be chosen* to give much smaller total differences of optical path than those due to the primary spherical aberration alone if its sign is opposite to that of the \( C_4 \) term. The best focus is therefore not usually coincident with the paraxial focus.

**Relations between Optical Path Aberrations and the Geometrical Aberration**

General arguments of the foregoing type enable us to forecast the *kinds* of wave forms to be expected from axially symmetrical optical wave forms.

systems, without a complicated discussion of ray paths. It is useful, however, to be able to pass from the wave form to a prediction of the geometrical aberration. We will deal first with spherical aberration.*

Referring to Fig. 113, let $P$ be a point on an axially symmetrical wave front $APM$ of which the axis is $AB_p'$, and $B_p'$ is the paraxial focus; the broken line $ARQ$ represents the trace of a spherical reference surface drawn with $B_p'$ as centre; the point $R$ is on the ray through $P$, and the notation is otherwise similar to that of Fig. 111.

The very short arc $PL$ is drawn through $P$ with centre $B_p'$, and the short line $QLM$ cutting the reference surface in $Q$ and the wave front in $M$ would pass through $B_p'$ if produced. It will be seen that $LM$ will be closely equal to the extra gap, between reference surface and wave, associated with the point $Q$ as compared with the point $R$. If the extra optical path is $\delta W_y$, the geometrical equivalent gap is $\delta W_y/n'$.

In the limit, when $LP$ and $PM$ are so short as to be equivalent to straight lines,

$$\overset{\text{LPM}}{B_p'}\overset{\text{PD}}{P}$$

Equating the sines of these angles, and writing $T'$ for the interval $B_p'D$,

$$\frac{\delta W_y/n'}{PM} = \frac{T' \cos U'}{PB_p'}$$

If, however, the height of $R$ above the axis is $y$, and the extra height of $Q$ is $\delta y$, a very close approximation gives

$$PM \approx \delta y \sec U'$$

* Geometrical discussions of the type to be used here are useful to give an introductory picture, but if a rigorous discussion is essential analytical methods are to be generally preferred. Advanced treatises should be consulted.
Substituting in the equation above we therefore obtain
\[
\frac{\delta W_y}{\delta y} \simeq \frac{n'T'}{PB_p'}
\]
and in the limit when \(\delta W_y\) and \(\delta y\) are infinitesimal
\[
\frac{dW_y}{dy} \simeq \frac{n'T'}{PB_p'}
\]
However, if we write
\[
PB_p' \approx RB_p' - RP = l' - \frac{W_y}{n'}
\]
the equation then becomes
\[
\frac{dW_y}{dy} \simeq \frac{n'T'}{l' - W_y/n'}
\]
so that
\[
T' \simeq \frac{l'}{n'} \frac{dW_y}{dy} \left(1 - \frac{W_y}{l'n'}\right)
\]  \(\text{(7.09)}\)

Now in a case of an axially symmetrical wave front suffering only from spherical aberration the term \(W_y/l'n'\) will, as shown on p. 177, be represented by a series commencing with \(y^4\). The above equation becomes, including all approximations,
\[
T' = \frac{l'}{n'} \frac{dW_y}{dy} (1 - O(y^4))
\]  \(\text{(7.10)}\)
The differential coefficient \(dW_y/dy\) will be represented by a series starting with a term in \(y^3\). The product of this with terms of \(O(y^4)\) will be \(O(y^7)\) and this will be negligible if we are concerned only with the region in which primary aberrations are appreciable; the bracket term may therefore be then omitted from the last expression.

**General Transverse Aberrations**

In this case the investigation of the previous section has to be somewhat modified because the aberration function is not symmetrical with respect to the paraxial-type reference direction \(E'B_1'\) (Fig. 108) as shown by equation (7.08). Moreover we shall be concerned with both the \(y\) and \(z\) components of the aberration. Also it will generally be convenient to discuss the transverse aberration of the principal ray (distortion) independently, and then to measure the transverse aberration of other rays with respect to the intersection point of the principal ray—thus taking the principal ray as the standard reference direction. This means that the term \(3C_{11}h'^3\rho \cos \phi\) will disappear from the primary aberration term in (7.08), as
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well as certain other higher terms which need not concern us here. Considering the plane \( z = 0 \), the arguments of the foregoing section will still lead to equation (7.09), but we then see, from (7.08), that the expression for \( W \) will contain terms of order \( \rho^2 \), or \( O(y^2) \) in the case \( z = 0 \). Hence (7.10) will become

\[
T'_{v'} = \frac{l'}{n'} \frac{\partial W_{v'}}{\partial y} \{ 1 - O(y^2) \}
\]

but, as before, the calculation of the strictly primary transverse aberration terms (in which the sum of the powers of \( \rho \) and \( b' \) does not exceed three) will not be affected by the presence of the bracket term \( \{ 1 - O(y^2) \} \), which can therefore still be omitted. Discussion of the problem by the equations of solid geometry gives parallel equations—

\[
\begin{align*}
T'_{v'} &= \frac{l'}{n'} \frac{\partial W}{\partial y} \\
T'_{z'} &= \frac{l'}{n'} \frac{\partial W}{\partial z}
\end{align*}
\]

which are valid for the discussion of the primary aberration.

**Axial or Longitudinal Aberration**

Referring again to the case of Fig. 113 (spherical aberration), let the interval \( B_v B_{v'} \) be denoted by \( A' \), and called the axial aberration. We then see that if \( A' \) and \( T' \) are taken as numerically positive

\[
\frac{A'}{T'} \approx \frac{l' - A'}{y}
\]

from which, using (7.05) and (7.10), it may readily be shown that in cases where small aberrations are involved

\[
A' = l'^3 \{ 4a_4 y^2 + O(y^4) \}
\]

The axial aberration is not, however, a generally suitable measure of the aberration. When the conjugate distances are great, an aberration small in terms of path difference may be very great when expressed as an axial intercept.

Laws of axial aberration, however, facilitate the discussion of the intersection near the focus of the rays from the various zones of the system. In the case of primary spherical aberration (see Fig. 114), the bundle of rays can be shown to pass through a region of maximum constriction (the so-called “disc of least confusion”) at a
distance from the paraxial focus three quarters of the distance towards the axial intersection point of the extreme marginal rays.\footnote{See Heath, \textit{Geometrical Optics}, 2nd edn. (Cambridge: University Press) p. 114.} The envelope of the aberrant rays is known as a "caustic."

\textbf{Geometrical Interpretation of the Primary Aberrations}

Transforming (7.08) into Cartesian coordinates by putting

\[ \rho \cos \phi = y, \text{ and } \rho^2 = y^2 + z^2 \]

we have

\[ W = aC_{40}(y^2 + z^2)^2 + 1C_{31}h'(y^2 + z^2)y + 2C_{22}h'^2y^2 \\
+ 2C_{20}h'^3(y^2 + z^2) + 3C_{11}h'^3y \]

whence, from (7.11),

\[ T'_y = (l'/n') \left\{ aC_{40}(y^2 + z^2)y + 1C_{31}h'(3y^2 + z^2) \\
+ 2C_{22}h'^2 \cdot 2y + 2C_{20}h'^2 \cdot 2y + 3C_{11}h'^3 \right\} \]

\[ T'_z = (l'/n') \left\{ aC_{40}(y^2 + z^2)z \\
+ 1C_{31}h' \cdot 2yz + 2C_{20}h'^2 \cdot 2z \right\} \] \hspace{1cm} (7.13)

The aberrations are more readily interpreted by transforming back\footnote{The expression could, of course, have been obtained directly from (7.10) by differentiation in polar coordinates, but both (7.13) and (7.14) are useful.} to polar coordinates thus—

\[ T'_y = (l'/n') \left\{ aC_{40}\rho^3 \cos \phi + 1C_{31}h'\rho^2(2 + \cos 2\phi) \\
+ 2C_{22}h'^2 \cdot 2\rho \cos \phi + 2C_{20}h'^2 \cdot 2\rho \cos \phi + 3C_{11}h'^3 \right\} \]

\[ T'_z = (l'/n') \left\{ aC_{40}\rho^3 \sin \phi \\
+ 1C_{31}h'\rho^2 (\sin 2\phi) + 2C_{20}h'^22\rho \sin \phi \right\} \] \hspace{1cm} (7.14)

The expression can now be discussed term by term. The primary \textit{spherical aberration} displacements (coefficient $aC_{40}$) show a radial displacement of the ray in the paraxial focal plane proportional to $\rho^3$. 
but the intersection point rotates round the paraxial focus with the rotation of the point of origin R. The effect of all the rays from the exit pupil would therefore give a patch of greater and greater size as the exit pupil is enlarged (Fig. 115). Works on Technical Optics should be consulted for fuller details.

In the case of coma (coefficient $C_{31}$) the rays from a zone of radius $\rho$ in the spherical reference surface are distributed round a circle of which the centre is radially displaced (in the $y$-direction) by a distance of $2kh'\rho^2$ where $k = (l'/n')_1 C_{31}$, from the paraxial image point $B_1'$. The radius of the circle is half the displacement, i.e. $kh'\rho^2$. Thus the rays from the various zones fall into a comet or balloon-shaped figure containing the circles of ray-intersection from the various zones. Since the cosine and sine of $2\phi$ appear in the formulae for the transverse displacements in this coma term, the point of intersection of a ray derived from a point with $\phi = 45^\circ$, say, intersects the image at the extremity of a radial line inclined at $90^\circ$ to the plane of symmetry ($XY$). The ray density increases rapidly towards the "head of the comet" (H, Fig. 116).

The terms with coefficients $2C_{22}$ and $2C_{20}$ jointly comprise the astigmatism and curvature of the field; let us write these coefficients as $c$ and $d$ for brevity, the $y$ and $z$ displacements can then be written,

\[
y = \text{constant} \times h'^2\rho(c + d) \cos \phi \]
\[
z = \text{constant} \times h'^2\rho(d) \sin \phi
\]

The rays derived from a circular locus in the reference surface therefore intersect the image plane in an ellipse with one axis of length proportional to $(c + d)$ in the radial direction and one of length correspondingly proportional to $d$ in the direction perpendicular (the sagittal direction). The size of the ellipse is proportional to $h'^2\rho$, but unless the signs and magnitudes of $c$ and $d$ are known it will not be
possible to determine whether the major axis will lie in the radial direction or the sagittal. The consequences of the elliptical intersection locus of rays derived from a circular zone were examined by Sturm. In particular, it can be shown that all the rays will pass through two mutually perpendicular focal lines. Fig. 117 shows only the typical course of the rays derived from the extremities of the tangential and sagittal diameters of the reference surface. When the size of the ellipse is very small in comparison with the diameter of the zone of the reference surface, the relative distances from the focal surface of the tangential line and the sagittal line will be

![Diagram of focal lines]

**Fig. 117.** Formation of "focal lines." If a screen is held in the region of the focus the illuminated patches will be short lines as suggested at T and S, with an ellipse at E.

obviously proportional to \((c + d)\) and \(d\) respectively, and independent of \(p\), though they are both proportional to \(k^2\), so that they lie (see p. 36) on independent curved surfaces symmetrical about the axis of symmetry of the system but touching at their poles at \(B_p\), the paraxial focal point on the axis. The "disc of least confusion," or greatest constriction of the rays, lies half-way between the focal lines. This can also be shown to correspond to the best physical focus.

More detailed analysis of optical systems shows that there will be a fundamental "curvature of the field" called the Petzval curvature (after J. Petzval who investigated the subject) and this curvature will persist, in general, even when the astigmatism is so corrected that the focal lines coincide; the term "curvature of field" commonly employed in optics thus has a significance other than the curvature of the surfaces containing the focal lines. The common terminology
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does not correspond exactly to the terms in the Hamiltonian expression (7.14).

The final primary aberration (coefficient $\gamma C_{11}$) is distortion. When expressed as a ray displacement it is independent of $\rho$ and $\phi$, and gives a shift of the principal ray intersection, in a radial direction only, away from the Gaussian or paraxial-type image point, by an amount proportional to $h^3$ (see expression (7.13)). The most obvious consequence is the disturbance of the similarity between the shapes of objects and their images situated in the conjugate planes. If the displacement is outwards the distortion is known as "pincushion," if inwards as "barrel" type (see Fig. 118).

![Image with pincushion distortion](image)

**Fig. 118. Distortion.**

**Primary Aberrations of a Thin Lens**

The foregoing account of the aberrations of a symmetrical optical system was based on considerations so general that the relative magnitudes of the terms in the formulae could not be given detailed examination. The analytical task of determining the explicit expressions, even for the primary aberrations of a thin lens, is quite considerable, and will not be set down here, but it is not difficult to see in an approximate way how they arise in terms of ray paths, etc.

(i) *Spherical Aberration*. This aberration may effect the images of all points in the field including the axial point. Consider a plano-convex lens (Fig. 119) refracting rays parallel to the axis. It was shown (see Fig. 10) that perfect ray union could be produced in such a case by using a suitable hyperboloidal surface. If a spherical surface is used, the curvature of the latter does not, like that of the hyperboloid, diminish away from the axis; the gap between the two increases at first (see p. 20 for the type of argument required) proportionally to the fourth power of the height of the zone; and there will be a corresponding fourth-power term in the expression for the optical path differences of the disturbances arriving in the paraxial focus; as the aperture increases, sixth power and even higher terms may become significant. The slope of the spherical
surface is too great, with increasing aperture, by an amount depending at first on the cube of the zonal radius; correspondingly the rays are refracted too far, and begin to cut the focal plane below the axis by a distance similarly proportional to \( y^3 \), though this only represents the first term in a series in odd powers of \( y \). The higher terms become significant as \( y \) increases.

![Diagram of a sphere and hyperboloid with a ray refracted at the sphere.](image)

**Fig. 119.**

The deviation produced by a thin prism is a minimum when the rays have equal angles of incidence and final refraction. A thin lens is said to be "bent" when the radii \( r_1 \) and \( r_2 \) of its surfaces are mutually changed, but so as to keep a constant "total curvature."

![Diagram of a crossed lens with spherical surfaces exhibiting minimum spherical aberration.](image)

**Fig. 120.** Crossed lens with spherical surfaces exhibiting minimum spherical aberration.

i.e. \( (1/r_1 - 1/r_2) \), and thus a constant focal length. Thus if a lens with spherical surfaces is bent into a shape (Fig. 120) that for a given zone there is such a symmetrical passage of the ray, there will be (very near this condition) a minimum amount of angular spherical aberration, though indeed it will not be removed completely by any possible bending with spherical surfaces. For a lens made of crown glass the radius of curvature of the first face has to be about one sixth that of the second face to secure the minimum condition. In a similar way it will be understood that there must be a minimum of
aberration for a certain zone for such conjugate distances as produce the minimum deviation effect; for example if the lens is equi-convex, there will be minimum aberration if the conjugate distances are equal. A general expression for the optical path difference of the disturbances arising at the paraxial focus from zones of various height \( y \) was given above in (7.05). Putting \( n' = 1 \) for air it is

\[
W = a_4 y^4 + O(y^6)
\]

If the value of \( y \) is small enough the major part of \( W \) is due to the \( y^4 \) term alone. In the case of a thin lens the value of \( a_4 \) can be worked out as a function of \( F \), the power of the lens; \( L_1 \), the reciprocal of the object distance; \( R_1 \), the curvature of the first surface; and \( n \), the refractive index of the lens; it is found that

\[
a_4 = \frac{1}{4} \left[ F^3 \left( \frac{n}{n-1} \right)^2 + F^2 L_1 \left( \frac{3n+1}{n-1} \right) - F^2 R_1 \left( \frac{2n+1}{n-1} \right) \right] \\
+ F L_1^2 \left( \frac{3n+2}{n} \right) - F R_1 L_1 \left( \frac{4n+4}{n} \right) + F R_1^2 \left( \frac{n+2}{n} \right)
\]

(7.15)

Thus it will be seen the spherical aberration follows a parabolic law in its variation when either \( L_1 \) or \( R_1 \) is the sole variable. The expression can take simpler forms* if other parameters are used.

(ii) Coma in a thin lens is generally encountered when, as in Fig. 121, the rays proceed towards an off-axis image point.

In the case illustrated the incident rays are parallel, but the ray above the centre meets the equivalent prism in the zone of the lens much more nearly in the symmetrical condition of transmission

(equal angles of incidence and final refraction) than the ray below; so that the latter has a relatively great deviation. The wave surface (normal to the rays) has therefore a section deviating from the spherical reference surface (centred on the principal ray at the point of intersection with the focal plane) in a manner suggested in the figure. The outer rays intersect above the principal ray, and the result can be shown to be the balloon-shaped figure illustrated more particularly in Fig. 116, the tail lying on the principal ray, and the extreme end of the figure lying at the crossing point of the marginal rays.

Equality of deviation of the upper and lower rays might clearly

![Diagram](image)

Fig. 122.

be attained by bending the lens until the final angle of refraction of one ray is equal to the angle of incidence of the other; this will not be far from the condition when coma disappears. It can be inferred that coma will also vary with the shape of the lens and the conjugate distances, but the relations can be shown to be linear instead of parabolic. Works on Technical Optics should be consulted for further details.

The effects of coma with a single positive lens used as in Fig. 121 are always partly masked by the other aberrations, especially the astigmatism.

**The Astigmatism of Oblique Pencils of Rays**

The oblique astigmatism arises in a thin lens from a general cause easily pictured in physical terms. The convergence of the wave front after refraction is due, of course, to the relative retardation of the disturbances passing through the thicker axial parts of the lens. But the central retardation (see Fig. 122) is the same both for the tangential and sagittal diameters in the oblique direction. If the
obliquity of the principal ray is \( \omega \), and the diameter of the lens is \( 2y \), the bundle of parallel rays entering the tangential diameter is only \( 2y \cos \omega \) in diameter, whereas the full diameter is operative for the extreme sagittal rays. The relative curvatures imparted to the refracted wave front will be easily seen to have the ratio of unity to \( \cos^2 \omega \) in the two cases (see below, p. 190). In the present discussion, however, the tangential and sagittal conjugate distance relations (1.35) and (1.36) may be used to investigate the problem.

Let the external angle of incidence (Fig. 123) of a ray of small obliquity be \( \omega \), then the internal angle will be, sufficiently nearly, \( \omega/n \) where \( n \) is the refractive index of the lens in air; and if the lens is very thin the internal and external angles will be sensibly equal at both faces. Thus two successive applications of the sagittal formula (1.35) yield

\[
\frac{n}{s_1' - s_1} = \frac{1}{r_1} \quad \frac{n}{s_2' - s_2} = \frac{1}{r_2} \quad \frac{n \cos (\omega/n) - \cos \omega}{\cos \omega - n \cos (\omega/n)}
\]

But if the lens has negligible thickness \( s_2 \to s_1' \), and thus by addition of the two above equations we find for the sagittal power \( F_s \)–

\[
F_s = \frac{1}{s_2'} - \frac{1}{s_1} = \left( n \cos (\omega/n) - \cos \omega \right) \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

Provided that \( \omega \) is very small, the value of the cosine will be represented sufficiently nearly by the first two terms in the usual expression, i.e.

\[
\cos \omega = 1 - \frac{\omega^2}{2} + \text{(negligible terms)}
\]

and

\[
\cos (\omega/n) = 1 - \frac{\omega^2}{2n^2} + \text{(negligible terms)}
\]

Also if \( F \) is the axial power of the lens,

\[
\left( \frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{F}{n - 1}
\]

Substituting from the last three expressions in (7.16), a little algebraical reduction leads to

\[
F_s \simeq \left( 1 + \frac{\omega^2}{2n} \right) F
\]
The tangential power \( F_t \) may be found in a very similar way by two applications of (1.36). Then we find

\[
F_t = \frac{1}{t_2} - \frac{1}{t_1} = \frac{n \cos (\omega/n) - \cos \omega}{\cos^2 \omega} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

whence \( \frac{F_s}{F_t} = \cos^2 \omega \)

It may be left as an exercise for the student to show that

\[
F_t \simeq \left( 1 + \frac{2n + 1}{2n} \omega^2 \right) F
\]

Fig. 124. T is trace of tangential focal surface. S is trace of sagittal focal surface.

The reciprocals of the tangential and sagittal powers represent the distances of the foci for the tangential and sagittal rays, where the tangential and radial focal lines (Fig. 117) are situated.

Over the image field we therefore find (as far as primary aberrations are concerned) one curved surface containing the sagittal lines and another containing the tangential lines. In the case of a single positive lens of glass passed centrally by the principal ray their approximate relative dispositions are shown in Fig. 124. Note from p. 184 that the length of a short focal line representing a star image will be proportional to \( \rho \) the radius of the aperture of the lens, and proportional to \( h^2 \), i.e. to the square of the distance of the image point above the axis.

Suppose that a lens of small aperture (working with equal conjugate distances so that the image and object sizes are nearly equal)
forms an image of an object field containing point sources of light (Fig. 125(a)). The images in the sagittal field then appear as radial lines, as in (b), and those in the tangential field as tangential lines, as in (c). If an object of the type (d) is used, each point of a radial line in the object will be imaged as a short radial image line in the sagittal field, and the overlapping of all such image elements allows the image of the whole line to appear sharply defined; but the image of the circular lines in (d) is diffuse, as in (e). The converse occurs in the tangential field, where the circular image lines appear sharp and the radial ones diffuse, as in (f). Such an object as (d) is therefore a convenient one for testing for the presence of astigmatism and finding the curvature (with suitable auxiliary apparatus) of the astigmatic image fields.

The discussions above have all concerned the case of a lens traversed centrally by the principal ray. If, however, a circular diaphragm is placed symmetrically on the axis but away from the
lens (Fig. 126) the principal ray will pass eccentrically through the latter and the astigmatism is profoundly changed. If the lens has a suitable meniscus shape concave to the diaphragm, the astigmatism may be removed or over-corrected, but in the case of correction the field of sharp focus (the Petzval surface) still remains curved; the radius of curvature of this image field for a thin lens can be shown to be about \( n f' \) where \( n \) is the refractive index of the lens.

**Distortion**

Provided that the principal ray traverses the centre of the thin lens, the surfaces encountered will be so nearly parallel that the deviation of the ray will be negligible, and there will be no distortion; but again, if the lens is used with a diaphragm at an appreciable distance from it the deviation of the principal ray (Fig. 127) is no longer (owing to the spherical aberration effect) strictly proportional to the incidence height, but will contain a term proportional to \( \omega^3 \) and thus to the cube of the image height (see p. 185). If the diaphragm is in front of the lens this deviation will be inwards, giving *barrel* distortion (Fig. 118); if behind, *pincushion* distortion will be found. The distortion is, however, obviously affected by the bending of the lens.

**Problems of Optical Design**

The more difficult problems of optical design are largely concerned with the avoidance of aberrations, both chromatic and monochromatic. It has been shown how axial chromatic aberration can be avoided in a telescope objective by the use of a negative lens of flint glass together with the positive crown lens. These lenses have, in general, spherical aberration of opposite sign which varies with the shape in each case; they also have coma of opposite sign, again
varying with the shape, although not at the same rate. It is accordingly often possible to choose such shapes for both the lenses that the spherical aberration and coma are compensated (although not the astigmatism, see p. 188).

Suppose further that the primary spherical aberration ($\rho^4$) contributions from the crown and flint cancel each other, but there are residual higher components ($\rho^6$, etc.) which are not balanced; if the lens is properly designed the residual higher terms may be made relatively harmless up to a higher aperture by leaving a residual amount of primary aberration of opposite sign so that the aberrations balance to some extent as suggested in Fig. 128.

This kind of balance is of very general application, so that the calculation of the higher aberration effects, and not merely the primary terms, has to be undertaken. Hitherto the amounts of the higher terms have generally been found by ray tracing, etc., rather than by analytical methods, but the development of new methods is still being intensively studied.*

* See Buchdahl, Optical Aberration Coefficients (Oxford: University Press).
1. A wave of spherical form is proceeding towards an axial focus point, the extreme associated rays making an angle of 10° with the axis. The radius of curvature of the wave on leaving the last limiting diaphragm is 20 cm. Calculate the displacements from the focus, (a) axial, (b) perpendicular to the axis, which will involve optical path differences reaching the Rayleigh limit. \( \lambda = 0.5 \times 10^{-4} \text{ cm} \).

*Hint: See Appendix II, and page 124.*

2. If the optical path differences near the paraxial focus of a wave are represented (with sufficient accuracy) by

\[
W = C_1 y^2 + C_2 y^4
\]

where \( y \) is the zonal radius with a maximum value unity, and \( C_1 \) is proportional to the axial displacement from the true paraxial focus, investigate the conditions when \( C_1 = -C_2 \). Find the maximum residual value of \( W \) in comparison with the value for the paraxial focus, and also the zonal radius for which the maximum occurs. Consider further the relative amounts of the maximum residuals for other values of \( C_1 \).

3. Find by the use of suitable tables the numerical error (to five figures) of the formula (see p. 175).

\[ \cos U' \approx 1 - y^2/2l^2 \]

for the cases when \( U' \) is 2° and 10° respectively, and the value of \( l' \) is 20 units.

4. A wave afflicted by spherical aberration has such a shape that (with the usual terminology)

\[ W = ay^4 \]

and the value of \( a \) is such that \( W = 2\lambda \) for the marginal value of \( y \), i.e. 2·0 cm.

If the paraxial radius of the wave is 16·0 cm, and the medium is air, find the primary transverse and axial aberrations of the marginal ray with reference to the paraxial focal plane. *N.B. \( \lambda = 0.5 \times 10^{-4} \text{ cm} \).*

5. An image field afflicted with primary coma shows the following type of ray aberration with respect to the crossing point of the principal ray in the focal surface—

Radial displacement \( = Ch\rho^2 (2 + \cos 2\phi) \)

Transverse displacement \( = Ch\rho^3 (\sin 2\phi) \)

where \( \rho \) and \( \phi \) represent the polar coordinates of the ray origin in the reference surface, \( h' \) is the image height, and \( C \) is a constant. If the maximum radial displacement for a ray is 0·2 mm when \( h' = 1 \text{ cm} \), and \( \rho \) is 1 cm, draw the complete coma figure to scale for the case \( h' = 0.5 \text{ cm} \), and \( \rho = 0.5 \text{ cm} \).

6. Show that the centre of the primary coma circle (in the focal surface), derived from a zone of the reference surface given by \( \rho = \text{constant} \), is equidistant in space from all points in the corresponding zone of the wave surface, supposing the latter to be free from aberration except for primary coma.

7. Find approximate expressions for the radii of curvature at the axial point of the sagittal and tangential surfaces for a thin lens, assuming the object field to be at infinity.

8. The radius of the so-called Petzval surface of a thin lens being given by \(-f'\rho \), where \( n \) is the refractive index, show that, measured along the principal ray, the tangential surface is three times as far from the Petzval surface as is the sagittal.
9. An image point is afflicted only with primary coma and astigmatism. Given the following values of the coefficients in equation (7.14)—

\[ sC_{21} = 0.04; \quad sC_{22} = sC_{30} = 0.01 \]

plot the aberration figure to scale, for the complete zone \( \rho = 1 \), for unit value of \( h' \).

10. If a quarter-plate camera is used to give a final picture measuring 3 in. by 4 in., and the image of the vertical edge of a building taken with the camera axis horizontal is found to touch the 4-in. edge of the picture in the centre, but (owing to barrel distortion) is within the margin by 0.1 in. at both top and bottom, find the coefficient \( C \) of primary distortion when the radial displacement of the image at a distance \( h' \) from the centre is \( Ch'^2 \).

11. Starting with the expression (7.15) for the spherical aberration of a thin lens, assume the following numerical values—

\[ n = 1.50, \quad F = 5.0 \]

Plot the coefficient \( a_4 \) for a series of values of \( R_1 \), assuming that the object is infinitely distant. Hence or otherwise find the shape of the lens giving minimum spherical aberration.

Further, assuming that \( R_1 = 5.0 \), plot the values of \( a_4 \) for a series of values of \( L_1 \), and hence or otherwise find the conjugate distances giving minimum spherical aberration.

12. An astigmatic eye needs a Cylindrical correction of + 0.5 D, Axis Horizontal. Show (a) that the cylindrical part of the correction could be given if the eye looks obliquely through a close lens of power + 5 D, with refractive index 1.5. Specify the position in which the lens must be held with respect to the direction of view.

Find also (b) the appropriate position and tilt of the lens if the object field is at infinity, and the vertex refraction of the eye, with reference to the vertical meridian, is + 6 D.
APPENDIX I

Laws of Refraction: Vector Forms

The vector form of the law of refraction is as follows: let \( s, s' \) be the unit vectors of the directions of the incident and refracted rays, and let \( o \) be the unit vector normal to the surface at the point of refraction. (Do not confuse the vector notation with that used for "reduced quantities" in Chapter II.) Denoting vectorial multiplication by \( \times \), Snell's Law becomes—

\[
(n's' - ns) \times o = 0
\]

This comprises both parts (p. 7) of the law.

Heath's Law follows directly. From the above

\[
n's' - ns = Co
\]

where \( C \) is a numerical factor. Let \( f \) be any unit vector tangential to the refracting surface at the point of incidence; then the scalar product \( o.f \) is zero. Hence on multiplying (A.1.1) by \( f \), in scalar fashion

\[
n's'.f - ns.f = 0
\]

Thus the incident and refracted rays form angles (\( \alpha \) and \( \alpha' \) respectively) with any line tangential to the surface at the point of incidence, such that

\[
n' \cos \alpha' - n \cos \alpha = 0
\]
APPENDIX II

Depth of Focus: Physical Discussion

Let AP (Fig. 129) represent the section of a symmetrical spherical wave front of radius $l$ converging towards B. The elementary disturbances (Huygens’s Principle) arriving at B from all parts of the wave front arrive in the same phase. The axial point $B_1$ is situated at an extra distance $\delta l$ along the axis, and the distance $AB_1$ now differs from $PB_1$. If the refractive index of the medium is $n$ it can be shown* that a difference in the concentration of light on the axis will be appreciable when $\delta l$ has so far increased that $n(AB_1 - PB_1) = \lambda/4$ where $\lambda$ is the wavelength of light. (This is the so-called “Rayleigh limit.”) Let $PB_1 = l_1$, then

$$AB_1 - PB_1 = l + \delta l - l_1$$

Now

$$l_1^2 = l^2 + (\delta l)^2 + 2l \cdot \delta l \cdot \cos U$$

(A.2.1)

where $U$ is the angle between the extreme ray and the axis; but

$$(l + \delta l)^2 = l^2 + (\delta l)^2 + 2l \cdot \delta l$$

(A.2.2)

Subtracting (A.2.1) from (A.2.2) we have

$$(l + \delta l)^2 = l_1^2 + 2l \cdot \delta l(1 - \cos U)$$

(A.2.3)

Let us write

$$\cos U = 1 - \frac{U^2}{2} + O(U^4)$$

Then from (A.2.3)

$$(l + \delta l)^2 = l_1^2 \left[ 1 + \frac{2l \delta l}{l_1^2} \left( \frac{U^2}{2} - O(U^4) \right) \right]$$

The expressions are exact as far as (A.2.3). The last one can


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be simplified with adequate accuracy if both \( \delta l \) and \( U \) are so small that \((\delta l)^2\) and \(U^4\) are negligible. In this case, taking square roots,

\[
 l + \delta l = l_1 \left( 1 + \frac{l \delta l}{l_1^2} U^2 \right)^{\frac{1}{2}} \\
= l_1 \left( 1 + \frac{1}{2} \frac{l \delta l}{l_1^2} U^2 + \text{negligible terms} \right)
\]

So that

\[
 l + \delta l - l_1 \approx \frac{1}{2} \frac{l \delta l}{l_1} U^2 
\]

(A.2.4)

The difference between \( l \) and \( l_1 \), when \( U' \) is not large, is obviously (from the diagram) of the same order as \( \delta l \). Hence,

\[
 l = l_1 - O(\delta l)
\]

and

\[
 \frac{l}{l_1} = 1 - O\left( \frac{\delta l}{l_1} \right) 
\]

(A.2.5)

Hence, by substituting from (A.2.5) in (A.2.4), we see that, within terms depending on \((\delta l)^2\),

\[
 \text{geometrical path difference} = l + \delta l - l_1 \approx \frac{1}{2} \delta l U^2 
\]

(A.2.6)

so that the optical path difference (O.P.D.) between disturbances from A and P\(_1\) arriving at B\(_1\) is

\[
 \text{O.P.D.} = \frac{1}{2} n \delta l U^2 
\]

(A.2.7)

Then if the O.P.D. may amount to \( \pm \lambda/4 \),

\[
 \delta l \leq \pm \frac{\lambda}{2nU^2} 
\]

(A.2.8)

The expression (A.2.8) was derived for the depth of focus of an image. However, it may be applied also to the discussion of the permissible movement of an object point (for example in the microscope) for which the image observed in a steady focal position will show no appreciable loss of definition.
APPENDIX III

Depth of Focus of a Lens in the Object Space

Let the optical system shown in Fig. 130 have an entrance pupil of diameter $a$, and let it form, on the screen $B'$, sharp images of all points in the plane B. Rays from a limited area (diameter $c$) of the plane B can reach the screen by passing through a point P in the plane C. Hence the point P will be imaged on the screen $B'$ by a ray patch corresponding to the diameter $c$ on plane B. If the magnification is $m$, the ray patch on $B'$ will have a diameter $mc$. We may specify that this shall not exceed some tolerable limit. If the distance between B and C is $\delta l_2$ while the axial distance from B to the entrance pupil is $l$, we shall have

$$\frac{\delta l_2}{c} = \frac{l - \delta l_2}{a} . \quad \cdots \quad (A.3.1)$$

Similarly, a point Q in a plane A can also be imaged as a patch not exceeding $mc$ if the axial distance $\delta l_1$ between A and B is given by

$$\frac{\delta l_1}{c} = \frac{l + \delta l_1}{a} . \quad \cdots \quad (A.3.2)$$

From these two equations we obtain

$$\delta l_1 = \frac{lc}{a - c} \quad \text{and} \quad \delta l_2 = \frac{lc}{a + c} . \quad \cdots \quad (A.3.3)$$

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The total depth of focus is

\[ \delta l_1 + \delta l_2 = \frac{2alc}{a^2 - c^2} \]

If the aperture \( a \) is made equal to the tolerable value of \( c \), then \( \delta l_1 \) is infinity, and the focal depth extends from infinity to \( l/2 \). The value of \( l \) depends of course on the back conjugate distance and the magnification.

The above discussion is of a strictly geometrical character; it will apply fairly well to the case of a photographic camera, but optical path considerations (see Appendix II) are necessary where the permissible aberrations are very small.
### APPENDIX IV

#### Refractive Index Data

<table>
<thead>
<tr>
<th>Type</th>
<th>$n_d$</th>
<th>$V$</th>
<th>$n_p - n_o$</th>
<th>$b - d$</th>
<th>$d - F$</th>
<th>$F - g$</th>
<th>$g - h$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
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<tbody>
<tr>
<td>Fluorite</td>
<td>1.43390</td>
<td>95.4</td>
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<td>0.00219</td>
<td>0.00917</td>
<td>0.00244</td>
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<td>1.0481</td>
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<td>0.536</td>
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<td>0.00859</td>
<td>0.00412</td>
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<td>0.00467</td>
<td>0.00856</td>
<td>1.480</td>
<td>0.699</td>
<td>0.544</td>
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<td>Dense barium crown</td>
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<td>58.5</td>
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<td>0.00501</td>
<td>0.00727</td>
<td>0.00570</td>
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<td>1.479</td>
<td>0.695</td>
<td>0.545</td>
<td>0.451</td>
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<tr>
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<td>0.00485</td>
<td>0.00711</td>
<td>0.00561</td>
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<td>0.01005</td>
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<td>0.01910</td>
<td>absorbent</td>
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<td>0.712</td>
<td>0.608</td>
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APPENDIX V

Dispersion Formulae

The study of the physical basis of dispersion lies in the sphere of physical optics. The classical theory envisages the presence, in the dispersing medium, of groups of oscillators capable of full or partial resonance. If such natural oscillator frequencies in a medium are \( \nu_0 \) and \( \nu_1 \), the kind of relation expected between the refractive index \( n \) and the frequency \( \nu \) of a disturbance passing through the medium can be shown to be

\[
\frac{n^2 - 1}{\nu_0^2 - \nu^2} = \frac{A}{\nu_0^2 - \nu^2} + \frac{B}{\nu_1^2 - \nu^2}
\]

Put into terms of wavelength (\( \lambda \)) this leads to the form (Ketteler)

\[
n^2 = \frac{M_1}{\lambda - \lambda_1^2} - \frac{M_2^2}{\lambda^2 - \lambda^2}
\]

However, formulae of this type are not fully accurate for solid materials and it is often the case that empirical formulae can be used with equal or better success to give adequate results for interpolation over a restricted range of wavelength. For example Hartmann’s simple formula

\[
n = n_0 + C
\]

is often useful, but Conrady’s form,

\[
n = n_0 + \frac{A}{\lambda} + \frac{B}{\lambda^{3.5}}
\]

is usually found to be more accurate in practice.

The relative partial dispersions of most ordinary glasses have an approximate linear relation with the \( V \)-values, for example it will be found that, for many ordinary glasses,

\[
\frac{n_\lambda - n_d}{n_F - n_C} \simeq A_{\lambda,d} + B_{\lambda,d} V
\]

where \( A_{\lambda,d} \) and \( B_{\lambda,d} \) are functions of \( \lambda, d \) independent of the constitution of the glass. In many cases the formula will approximate to

\[
n_\lambda - n_d = A_\lambda(n_F - n_C) + D_\lambda
\]
where $A_\lambda$ and $D_\lambda$ vary little from glass to glass. It can be readily shown that when glasses conforming to such relations are used to form achromatic doublets, the secondary spectrum bears a fixed relation to the focal length.

**Hartmann Constants**

Typical values of the constants in the Hartmann formula above, suitable for calculations within the range of the visible spectrum, are as follows—

(Wavelengths are expressed in terms of millimicrons (m$\mu$), so that, for example, for the mean of the sodium lines $\lambda = 589.3$ m$\mu$.)

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<tr>
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<th>$n_d$</th>
<th>$V$</th>
<th>$n_o$</th>
<th>$C$</th>
<th>$\lambda$</th>
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<td>Ordinary crown glass</td>
<td>1.50649</td>
<td>62.2</td>
<td>1.48866</td>
<td>7.5485</td>
<td>166.02</td>
</tr>
<tr>
<td>Ordinary flint glass</td>
<td>1.62572</td>
<td>35.6</td>
<td>1.59259</td>
<td>12.474</td>
<td>212.89</td>
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APPENDIX VI

The Optics of Non-homogeneous Media

Let the medium be isotropic but not necessarily homogeneous; then the refractive index $n$ is a function of position in space; let it be defined by some continuous function, so that

$$n = f(x, y, z)$$

The path of a ray in such a medium will be such as to fulfil the relation (4.03) (p. 110) which can in this case be written

$$\delta fn \, ds = 0$$

(A.6.1)

Let the value of the integral between two fixed points $A$ and $B$ be written $V$, then if the path be subject to a small variation (p. 110)

$$\delta V = \int_A^B \delta n \, ds + \int_A^B n \delta(ds).$$

(A.6.2)

Now since

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds = \frac{dx}{ds} dx + \frac{dy}{ds} dy + \frac{dz}{ds} dz$$

and

$$\delta(ds) = \frac{dx}{ds} \delta(dx) + \frac{dy}{ds} \delta(dy) + \frac{dz}{ds} \delta(dz)$$

(A.6.3)

if the variation is so small that terms like $dx \delta(dx/ds)$ are inappreciable.

The second integral in (A.6.2) can now be written

$$\int_A^B \left\{ n \frac{dx}{ds} \delta(dx) + n \frac{dy}{ds} \delta(dy) + n \frac{dz}{ds} \delta(dz) \right\}$$

Since the signs of variation and differentiation are commutative, this integral becomes, on integrating by parts,

$$\left[ n \frac{dx}{ds} \delta x + n \frac{dy}{ds} \delta y + n \frac{dz}{ds} \delta z \right]_A^B$$

$$- \int_A^B \left\{ \frac{d}{ds} \left( n \frac{dx}{ds} \right) \delta x + \frac{d}{ds} \left( n \frac{dy}{ds} \right) \delta y + \frac{d}{ds} \left( n \frac{dz}{ds} \right) \delta z \right\} ds$$

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Since
\[ \delta n = \frac{\partial n}{\partial x} \delta x + \frac{\partial n}{\partial y} \delta y + \frac{\partial n}{\partial z} \delta z \]
the total value of \( \delta V \) in (A.6.2) is
\[
\delta V = \left[ n \left( \frac{d x}{d s} \delta x + \frac{d y}{d s} \delta y + \frac{d z}{d s} \delta z \right) \right]_A^B \\
+ \int_A^B \left[ \frac{\partial n}{\partial x} - \frac{d}{d s} \left( n \frac{d x}{d s} \right) \right] \delta x + \left[ \frac{\partial n}{\partial y} - \frac{d}{d s} \left( n \frac{d y}{d s} \right) \right] \delta y \\
+ \left[ \frac{\partial n}{\partial z} - \frac{d}{d s} \left( n \frac{d z}{d s} \right) \right] \delta z \right] ds = 0
\]

Now \( \delta V \) must vanish for all very small variations of the path so that the required conditions are that
\[
\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z = 0 \quad \ldots \quad (A.6.4)
\]
at each end; and that further, all along the ray,
\[
\frac{d}{ds} \left( n \frac{dx}{ds} \right) - \frac{\partial n}{\partial x} = 0 \quad \ldots \quad (A.6.5)
\]
\[
\frac{d}{ds} \left( n \frac{dy}{ds} \right) - \frac{\partial n}{\partial y} = 0
\]
\[
\frac{d}{ds} \left( n \frac{dz}{ds} \right) - \frac{\partial n}{\partial z} = 0
\]
so that the integration of any two of these last equations will suffice to determine the path of a ray.

Note that, in equation (A.6.4), \( dx/ds \) is the direction cosine of the ray, while \( \delta x \) is proportional to the direction cosine of the interval representing the variation of the path. If then we suppose that A and B, the terminal points, move on surfaces between which the optical path \( V \) is constant, the rays along which this optical path are measured must be normal to the surfaces at each end (orthotomic property).

The three equations (A.6.5) are analogous to the Euler–Lagrange equation of mechanics. By Newton’s Theorem “the path of a ray in any medium is identical with that of a particle moving freely with velocity \( n \) under the action of a conservative system of forces.”

A principle of “least action” was cited by Maupertuis, during the eighteenth century, in the supposition that, in all natural processes, the “action” measured by the product of the path, time and velocity must be a minimum.
A modern statement of the principle is contained in the equation

\[ \delta A = \delta \int_M^N mv \, ds = 0 \]

\( A \) is the "action" involved in the movement of a particle of mass \( m \) along a path in space between two points \( M \) and \( N \); that is, the line integral of the product of the momentum and the element of path \( ds \). The "variation" of the action refers to any other possible path between \( M \) and \( N \), associated with the first in the same way as described for the varied optical path. The movement takes place under the action of a "conservative system of forces" so that the total energy remains constant. Then the "mechanically possible" or unconstrained path will be such as to fulfil the above equation; any small deviation from it will produce only a relatively negligible change in the value of \( A \). The variation of the path of a particle in space might be imagined to be produced by an imaginary frictionless tube, connecting \( M \) and \( N \), which could be slightly altered in shape. Motion in the tube would not involve any loss of energy in friction.

The above equation led to the formulation of Hamilton's principle which concerns the "variation" of the motion of a system of free particles moving in a conservative system of forces. If \( L \) is the kinetic energy and \( V \) the potential energy of the system, Hamilton's Principle states that

\[ \int_{t_0}^{t_1} \delta(L - V) \, dt = 0 \]

where the position and relative configuration of the system is unvaried at the times \( t_0 \) and \( t_1 \) corresponding to the limits of integration. If the potential \( V \) is a function of \( S \) generalized and mutually independent coordinates \((q)_i\), and \( L \) is a function of the coordinates and their time-derivatives, the generalized equations of motion (Lagrange) become

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = -\frac{\partial V}{\partial q_i} \quad (i = 1 \text{ to } S) \]

In the case of a single particle of mass \( m \) moving in a Cartesian system, \( L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \), so that the partial derivative \( \partial L/\partial x \) = 0, and the Lagrange equations reduce to

\[ m \frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \]

with the parallel equations for \( y \) and \( z \).

Just as in the optical case the ray paths are orthotomic to the wave surfaces, the mechanically possible paths are orthotomic to surfaces of "action," or equal values of \((L - V)\) (called by Helmholtz the "kinetic potential").
APPENDIX VII

Refraction of Rays enclosing an Element of Solid Angle

The plane XZ in Fig. 131 separates media having refractive indices \( n \) and \( n' \) respectively. AO is a ray in the medium \( n \) incident at O with an angle (\( \widetilde{AOY} \)) of incidence \( i \). It is refracted into the direction OB with angle of refraction \( i' \).

Let the plane of incidence AOY containing both incident and refracted rays rotate through a very small angle \( d\theta \) around the axis OY. Dropping a perpendicular AP from A to the \( y \)-axis it is readily seen that the linear movement of A consequent on the rotation is

\[
AP \, d\theta = OA \sin i \, d\theta
\]

Similarly the linear movement of B will be \( OB \sin i' \, d\theta \). These small movements are sensibly perpendicular to the plane of incidence.

Further, a movement of the ray AO through a small angle \( di \) in the plane of incidence will be accompanied by a corresponding movement \( di' \) of the refracted ray, found immediately by differentiating the law of refraction; thus

\[
n \cos i \, di = n' \cos i' \, di'
\]  \hspace{1cm} \text{(A.7.1)}

By the combination of these movements, the point A would trace a small area (perpendicular to the ray) of amount

\[
OA \sin i \, d\theta \cdot OA \, di = OA^2 \sin i \, d\theta \, di
\]

and the solid angle \( d\omega \) subtended at O will thus be

\[
d\omega = \sin i \, d\theta \, di
\]

The corresponding solid angle for the refracted ray will be

\[
d\omega' = \sin i' \, d\theta \, di'
\]
Hence
\[ \frac{d\omega}{d\omega'} = \frac{\sin i \, di}{\sin i' \, di'} = \frac{n' \, di}{n \, di'} \]

Substituting from equation (A.7.1) above for \( di/di' \) gives the relation—
\[ n^2 \cos i \, d\omega = n'^2 \cos i' \, d\omega' \]
ANSWERS TO EXERCISES

CHAPTER I

1. $2,267 \pm 2$ miles.
2. 3 ft.
3. $\frac{dh \cos^2 \theta}{2d \cos \theta - h \sin \theta} + d \sin \theta$

5. Formula gives result about $1^\prime 55''$ too small. If angle of incidence on 1st face increases by 1', total deviation increases by about 5'.

6. 6.78 in. very nearly.

7. Condition is that radius of inner sphere is less than radius of outer sphere divided by $n$.

10. About 0.48 cm.
11. About 0.434 cm.

12. Sagittal distance $= s_0 (1 - \frac{4 n_i^2}{n})$; and axial radius $= s_0/(1 + n)$; where $s_0 = mn/(n - 1)$.

CHAPTER II

1. 13.5 mm approx.
2. 85.9 cm approx.
3. $f \simeq -16.65$ mm; $f' \simeq 22.2$ mm; separation of images $\simeq 0.291$ mm.

4. Assuming that the image-space is in the liquid, and that A and B are the poles of the surface, $AF' = -32$ cm, $BF' = 14$ cm, $AP = AP' = -8$ cm. The nodal points coincide at the centre of the sphere.

5. Distance of point of rotation from second principal focus is $-\{ -n xy/n \}^\frac{1}{2}$

8. $f \simeq -16.78$ mm; $f' \simeq 22.42$ mm; second principal point is approx. 1.75 mm inside cornea; second principal focus is approx. 24.17 mm from cornea.

9. Power of system is approx. 15.44 D; focal length is approx. 6.48 cm. Distance from front apex to principal focus $\simeq 5.44$ cm (outside), distance from front apex to principal point $\simeq 1.04$ cm (inside). (Both principal points coincide.)

10. The conjugate planes may be said to coincide in a lens of negligible thickness; alternatively a minimum separation is found when the conjugate distances are numerically equal and of opposite sign, i.e. $l' = -l = 2f'$.

12. $l' = \frac{1}{l}( -d \pm \sqrt{(d^2 + 4df')})$, so that $d^2 + 4df'$ must be positive if a real solution is to be found.

13. Both images virtual. Distances from surfaces are: tangential 3.43 cm approx., and sagittal 10.9 cm approx., measured along the principal ray.

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ANSWERS TO EXERCISES

CHAPTER III

1. \( \delta' \simeq 0.131 \text{ in.}; \) corresponding \( \delta \simeq 15.85 \text{ in.} \)

2. Crown Flint
   \[
   \begin{align*}
   f' & = 20.2 & -34.0 \\
   r_1 & = 20.8 & -20.7 \\
   r_2 & = 20.7 & \infty
   \end{align*}
   \]
   results given in cm only to the third significant figure.

4. All Gaussian points are at infinity when instrument is in “afocal” condition. When readjusted for accommodated eye \( f' = -f = 175 \text{ cm}; \) distance from objective to 1st principal plane \( \simeq 1,001 \text{ cm}; \) distance from eyepiece to 2nd principal plane is \( -200 \text{ cm}. \)

6. \( f_0 = 60 \text{ cm}; \ d = 40 \text{ cm}. \)

7. \( f' = 3/4; \ AP = \tfrac{1}{2}; \ AF = -\tfrac{1}{4}. \) The system is symmetrical about the mid-plane.

9. The focal length of the system is equal to \(-f'. \) The principal planes of the system are situated outside it on both sides at a distance \( 2f' \) from the nearest lens. The cross-wires may be placed at a distance \( f' \) in front of the first lens, or mid-way between the two last lenses.

11. The images formed by successive reflections at the two mirrors are coincident when the angle is \( 90^\circ. \) When the angle is changed by \( \theta, \) the two images subtend an angle of \( 4\theta \) with respect to the foot of the perpendicular drawn from the object point to the line of intersection.

13. \( F_2N = (f_1/f_0)(f_1 + f'_s); \ F_2'N' = -(f_0/f_0)(f_1 + f'_s). \) See also the formulae in No. 12.

CHAPTER IV

1. \( i' \simeq 47^\circ 3'. \)

3. \( 9x^4 + 18x^2y^2 + 9y^4 - 96x^3 - 96xy^2 - 176x^2 - 432y^2 + 384x = 0. \)

4. Approx. \( 2.23 \times 10^{-4} \text{ radians per millimicron}. \)

5. \( (J/\delta) \simeq 6770; \) Slit width \( \simeq 2.1 \times 10^{-4} \text{ cm}. \)

6. Approx. 0.0027 degrees in each case; result independent of distance.

7. Approx. 7,400.

9. \( r \simeq 3.94 \text{ cm}; \ n \simeq 1.538. \)

10. (a) The objective may be illuminated by light diffused from a diffusing medium such as a sheet of opal glass—oiled on if necessary.

   (b) The presence of air layers between the object plane and the objective would restrict the N.A. to unity.

CHAPTER V

1. Approx. 5.71 cm.

2. 164 cm.

3. Hypermetropia approx. + 3.14 D. Amp. of Ac. approx. 1.78 D.

6. Refraction error is approx. + 9.52 D; amplitude of accommodation approx. 9.52 D. When correction is worn, near point will be about 12.9 cm from the cornea.
GEOMETRICAL OPTICS

7. Eye is non-astigmatic; it is hyperopic (0.5 D) in the condition of
deviation and has 0.5 D amplitude of accommodation.

8. (a) 5.06 D S/– 2.47 D C Ax. 30°.
(b) 4.23 D S/6.30 D C Ax. 135°.

9. (a) 2.50 D S/2.25 D C Ax. 120°; 4.75 D C Ax. 120°/2.50 D C Ax. 30°.
(b) 2.0 D S/– 6.5 D C Ax. 45°; – 4.5 D C Ax. 45°/– 2.0 D C Ax. 135°.

10. Minimum magnifying power about 60. Required aperture approx.
6.8 cm.

11. About 0.5 mm diameter for the exit pupil.

12. Rather more than 5,000. Diffraction effects would prevent the restric-
tion of illumination to one cone.

CHAPTER VI

1. \[ 4\pi \mathcal{C}. \]

2. If \( r \) is the distance of a point in the screen from the foot of a perpendicu-
lar from the source, then \( E = (I/d^2) (1 + r^2/d^2)^{-3/2}. \)

3. Illumination under lamps is about 0.135 f.c., and rises to about 0.144 f.c.

4. \[ E = \frac{C}{d^2} \left[ \frac{l}{1 + r^2/d^2} + d \{\text{arc tan} (l/d)\} \right]. \]

5. \( B \simeq 0.618 \) stilb.

6. \( E \) is proportional to \( \cos^4 \theta \).

7. Illumination in exit pupil = \( IM^4 \).

9. If \( r_1 \) and \( r_2 \) are the respective maximum distances, then \( r_1/r_2 = d_1/d_2. \)

10. \((Lp/\pi) \cos \theta\).

11. \((F/4\pi^2) \cdot \rho/(1 - \rho)\).

CHAPTER VII

1. Axial displacement approx. 0.008 mm; transverse displacement approx.
0.0036 mm.

2. Maximum residual is one-quarter that found at paraxial focus. Zonal
radius for maximum = \( 1/\sqrt{2} \).

3. For \( U' = 2^\circ \), error is negligible; for \( U' = 10^\circ \), tables give 0.98481,
formula 0.98492.

4. \( T' \simeq 0.032 \) mm; \( L' \simeq 0.256 \) mm.

7. Radius of sagittal field = \( f' n/(n + 1) \).
Radius of tangential field = \( f' n/(3n + 1) \).

10. \( C \simeq 0.0097 \).

11. Aberration minimum when \( R_1 \simeq 8.57, R_2 = -1.43 \). For lens of the
fixed shape, aberration minimum when \( L_1 = -2.5 \) (equi-conjugate position).
(It is also zero when \( l_1 = 0. \))

12. (a) Lens is to be tilted from normal position about 18° round the
horizontal axis if held close to eye.
(b) The required distance between the lens and the eye will be about
1.2 cm, and the necessary tilt will be about 17°.
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